I will first formulate a problem in the theory of group representations and show how to solve it; then I will discuss the relation of this problem to the theory of automorphic forms. Since there is no point in striving for maximum generality, I start with a connected semisimple group $G$ with finite center. An irreducible unitary representation $\pi$ of $G$ on the Hilbert space $H$ is said to be square-integrable if for one and hence, as one can show, every pair $u$ and $v$ of nonzero vectors in $H$ the function $(\pi(g)u, v)$ is square-integrable on $G$. It is said to be integrable if for one such pair $(\pi(g)u, v)$ is integrable.

Suppose $\Gamma$ is a discrete subgroup of $G$ and $\Gamma\backslash G$ is compact. As was shown by Godement in an earlier lecture the representation $\pi$ of the previous paragraph occurs a finite number of times, say $N(\pi)$, in the regular representation on $L^2(\Gamma\backslash G)$. The problem is first to find a closed formula for $N(\pi)$. The method which I will now describe of obtaining such a formula is valid only when $\pi$ is actually integrable.

Square integrable representations are similar in some respects to representations of compact groups; in particular they satisfy a form of the Schur orthogonality relations. There is a constant $d_\pi$ called the formal degree of $\pi$ such that if $u', v', u$, and $v$ belong to $H$ then

$$\int_G (\pi(g)u', v')(\pi(g)u, v) \, dg = d_\pi^{-1}(u', u)(v, v').$$

If $u$ and $v$ are such that $(\pi(g)u, v)$ is integrable and $\pi'$ is unitary representation of $G$ on $H'$ which does not contain $\pi$, then

$$\int_G (\pi'(g)u', v')(\pi(g)u, v) \, dg = 0$$

for all $u', v'$ in $H$.

Let $L_i$, $1 \leq i \leq N(\pi)$, be a family of mutually orthogonal invariant subspaces of $L^2(\Gamma\backslash G)$ which are such that the action of $G$ on each of them is equivalent to $\pi$. Suppose that $\pi$ does not occur in the orthogonal complement of

$$\bigoplus_{i=1}^{N(\pi)} L_i.$$

If $\pi$ is integrable there is a unit vector $v$ in $H$ such that $(\pi(g)v, v)$ is integrable. Let $v_i$ be a unit vector in $L_i$ corresponding to $v$ under some equivalence between $H$ and $L_i$. The
orthogonality relations imply that the operator $\Phi \rightarrow \Phi'$ with
\[
\Phi'(g) = d_\pi \int_G \Phi(gh)(\pi(g)v,v)\,dh,
\]
\[
= \int_{\Gamma\backslash G} \Phi(h)\left\{ \sum_\Gamma \xi(g^{-1}\gamma h) \right\} \,dh,
\]
if $\xi(g) = d_\pi(\pi(g)v,v)$, is an orthogonal projection on the space spanned by $v_1, \ldots, v_{N(\pi)}$. For our purposes it may be assumed that $v$ transforms according to a finite-dimensional representation of some maximal compact subgroup of $G$. Then the argument used by Borel in a previous lecture shows that
\[
\sum_\Gamma \xi(g^{-1}\gamma h)
\]
converges absolutely uniformly on compact subsets of $G \times G$. Hence $v_1, \ldots, v_{N(\pi)}$ may be supposed continuous. As a consequence
\[
\sum_{i=1}^{N(\pi)} v_i(g)\bar{v}_i(g) = \sum_\Gamma \xi(g^{-1}\gamma h).
\]

Set $h = g$ and integrate over $\Gamma \backslash G$ to obtain
\[
N(\pi) = \int_{\Gamma\backslash G} \sum_\Gamma \xi(g^{-1}\gamma g)\,dg.
\]

The sum in the integrand may be rearranged at will. If $\Sigma$ is a set of representatives for the conjugacy classes in $\Gamma$ the integral on the right equals
\[
\int_{\Gamma\backslash G} \sum_{\gamma \in \Sigma} \sum_{\delta \in \Gamma_\gamma \backslash \Gamma} \xi(g^{-1}\delta^{-1}\gamma \delta g)\,dg = \sum_{\gamma \in \Sigma} \int_{\Gamma_\gamma \backslash G} \xi(g^{-1}\gamma g)\,dg
\]
\[
= \sum_{\gamma \in \Sigma} \mu(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g)\,dg,
\]
if $\Gamma_\gamma$ and $G_\gamma$ are the centralizers of $\gamma$ in $\Gamma$ and $G$ respectively. The equality of $N(\pi)$ and the final expression is of course a special case of a formula of Selberg and has been known for some time.

The problem of evaluating $\mu(\Gamma_\gamma \backslash G_\gamma)$, the volume of $\Gamma_\gamma \backslash G_\gamma$, has been discussed in the lectures on Tamagawa numbers. So we shall not worry about it now. Since $\Gamma \backslash G$ is compact every element of $\Gamma$ is semisimple; thus our problem is to express the integral
\[
\int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g)\,dg
\]
in elementary terms when $\gamma$ is a semisimple element of $G$.

If $\pi$ is a square-integrable representation of $G$ on $H$, $v$ is a vector in $H$ which transforms according to a finite-dimensional representation of some maximal compact subgroup of $G$, and
\[
\xi(g) = d_\pi(\pi(g)v,v),
\]
then a recent theorem of Harish-Chandra states that
(a)\[
\int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g)\,dg
\]
exists for $\gamma$ semisimple and vanishes unless $\gamma$ is elliptic, that is, belongs to some compact subgroup of $G$. Since $\Sigma$ contains only a finite number of elliptic elements the sum in the expression for $N(\pi)$ is finite. We still require a closed expression for the integrals appearing in it.

Let $K$ be a maximal compact subgroup of $G$. Since $G$ has a square-integrable representation there is a Cartan subgroup $T$ of $G$ contained in $K$. It is enough to compute the integral (a) for $\gamma$ in $T$. There is a limit formula of Harish-Chandra which allows one to compute its value at the singular elements once its values at the regular elements are known. Thus we need only evaluate it when $\gamma$ is regular. It should be remarked that in this limit formula there is a constant which depends on the choice of Haar measure on $G\gamma$. The exact relation of this constant to the choice of Haar measure has never been determined; until it is, our problem cannot be regarded as completely solved.

If $\gamma$ is regular and the measure on $G\gamma$ is so normalized that the volume of $G\gamma$ is one, then

$$\int_{G\gamma\backslash G} \xi(g^{-1}\gamma g) \, dg = \chi_\pi(\gamma^{-1})$$

if $\chi_\pi$ is the character of $\pi$. An explicit expression for the right-hand side has recently been obtained.

Let $\mathfrak{h}$ be the Lie algebra of $T$; choose an order on the roots of $\mathfrak{h}_C$; and let $\Lambda$ be a linear function on $\mathfrak{h}_C$ such that $\Lambda + \rho, \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$, extends to a character of $T$ and so that $(\Gamma + \rho, \alpha) \neq 0$ for all roots $\alpha$. Assume, for simplicity, that $\rho$ also extends to a character of $T$. To each such $\Lambda$ there is a square-integrable representation $\pi_\Lambda$ and if $H \in \mathfrak{h}$

$$\chi_{\pi_{\Lambda}}(\exp H) = (-1)^m \epsilon(\Lambda) \sum_{\sigma \in W} \frac{\text{sgn } \exp(\sigma(\Lambda + \rho))(H)}{\prod_{\alpha > 0} \{\exp(\alpha(G)/2) - \exp(-\alpha(G)/2)\}}.$$

Here $m = \frac{1}{2} \dim G/K, \epsilon(\Lambda) = \text{sgn}(\prod_{\alpha > 0}(\Lambda + \rho, \alpha))$, and $W$ is the Weyl group of $K$. Every square-integrable representation is equivalent to $\pi_\Lambda$ for some $\Lambda$. However the values of $\Lambda$ for which $\pi_{\Lambda}$ is integrable are not yet known. For some special cases see [1] and [2].

The geometrical meaning of the numbers $N(\pi_{\Lambda})$ is not yet completely clear. I would like to close this lecture with some suggestions as to what it might be. Since the evidence at present is rather meagre, they are only tentative. If $\mathfrak{g}_C$ is the complexification of the Lie algebra of $\mathfrak{g}$, the elements of $\mathfrak{g}_C$ may be regarded as left-invariant complex vector fields on $G$ and $G/T$ may be turned into a complex manifold in such a way that the space of antiholomorphic tangent vectors at $\tilde{g} = gT$ is the image of $\mathfrak{n}_C$ if $\mathfrak{n}_C$ is the subalgebra of $\mathfrak{g}_C$ generated by root vectors belonging to negative roots. Let $V^*$ be the bundle of antiholomorphic cotangent vectors and introduce a $G$-invariant metric in $V^*$ and hence in $\bigwedge^q V^*$. Let $B$ be the line bundle over $G/T$ associated to the character $\xi(\exp H) = \exp(\Lambda(H))$ of $T$. If $\Gamma$ is a discrete subgroup of $G$ let $C^q(\Lambda, \Gamma)$ be the space of $\Gamma$-invariant cross-sections of $B \otimes \bigwedge^q V^*$ which are square integrable over $\Gamma\backslash G/T$. There is a unique closed operator $\partial$ from $C^q(\Lambda, \Gamma)$ to $C^{q+1}(\Lambda, \Gamma)$ whose domain contains the infinitely differentiable cross-sections of compact support on which $\partial$ is to have its usual meaning and whose adjoint is defined on the infinitely differentiable cross-sections of $C^{q+1}(\Lambda, \Gamma)$ with compact support.

Set $C^q(\Lambda, \{1\}) = C^q(\Lambda)$. I expect, although I do not know how to prove it, that when $\Lambda + \rho$ is nonsingular the range of $\partial$ is closed for every $q$. If this is so then the cohomology groups $H^1(\Lambda)$ will be Hilbert spaces on which $G$ acts. Is it true that they vanish for all but
one value of $q$, say $q = q_\Lambda$, and that the representation $\pi'_\Lambda$ of $G$ on $H^{q_\Lambda}(\Lambda)$ is equivalent to $\pi_\Lambda$? The following theorem is a clue to the value of $q_\Lambda$.

**Theorem (P. Griffiths).** Let $a_1$ be the number of noncompact positive roots for which $(\Lambda + \rho, \alpha) > 0$ and let $a_2$ be the number of compact positive roots for which $(\Lambda + \rho, \alpha) < 0$. There is a constant $c$ such that if $|(\Lambda + \rho, \alpha)| > c$ for every simple root, $\Gamma \backslash G$ is compact, and $\Gamma$ acts freely on $G/T$, then $H^q(\Lambda, \Gamma) = 0$ unless $q = a_1 + a_2$.

It is, I think, worthy of remark that if one assumes that $H^q(\Lambda) = \{0\}$ for $q \neq q_\Lambda = a_1 + a_2$, then a formal application of the Woods Hole fixed point formula shows that if $\gamma$ is a regular element of $T$, then the value at $\gamma$ of the character of $\pi'_\Lambda$ is $\chi_{\pi_\Lambda}(\gamma)$. By the way, it is known that $H^0(\Lambda) = 0$ unless $q_\Lambda = 0$ and that if $q_\Lambda = 0$ the representation of $G$ on $H^0(\Lambda)$ is in fact $\pi_\Lambda$.

Finally one will want to show that when $\pi_\Lambda$ is integrable and $\Gamma \backslash G$ is compact the number $N(\pi_\Lambda)$ is equal to the dimension of $H^{q_\Lambda}(\Lambda, \Gamma)$. This can be done when $q_\Lambda = 0$; in this case $H^0(\Gamma, \Lambda)$ is a space of automorphic forms.

It should be possible, although I have not done so, to test these suggestions for groups whose unitary representations are well understood, in particular, for $SL(2, \mathbb{R})$ and the De Sitter group. To do this one might make use of an idea basic to Kostant’s proof of the (generalized) Borel-Weil theorem for compact groups. Suppose $\sigma$ is a unitary representation of $G$ on a Hilbert space $V$. Let $C^q(V)$ be the space of all linear maps from $\wedge^q n_C$ to $V$. $C^q(V)$ is a Hilbert space. The usual coboundary operator from $C^q(V)$ to $C^{q+1}(V)$ can be defined on those elements of $C^q(V)$ which take values in the Gårding subspace of $V$. The closure $d$ of this operator is the adjoint of the restriction of its formal adjoint to those elements of $C^{q+1}(V)$ which take values in the Gårding subspace. Of course $T$ acts on $\wedge^q n_C$. If $f \in C^q(V)$ define $tf = f'$ by $f'(X) = tf(t^{-1}X), X \in \wedge^q n_C$. There is a natural identification of $C^q(\Lambda)$ with the set of $f$ in $C^q(L^2(G))$ such that $tf = \exp(-\Lambda(H))f$ if $t = \exp H$ belongs to $T$ and of $C^q(\Lambda, \Gamma)$ with the set of $f$ in $C^q(L^2(\Gamma \backslash G))$ such that $tf = \exp(-\Lambda(H))f$. Moreover the following diagrams are commutative.

\[
\begin{array}{ccc}
C^q(\Lambda) & \xrightarrow{\bar{\delta}} & C^{q+1}(\Lambda) \\
\downarrow & & \downarrow \\
C^q(L^2(G)) & \xrightarrow{d} & C^{q+1}(L^2(G))
\end{array}
\quad
\begin{array}{ccc}
C^q(\Lambda, \Gamma) & \xrightarrow{\bar{\delta}} & C^{q+1}(\Lambda, \Gamma) \\
\downarrow & & \downarrow \\
C^q(L^2(\Gamma \backslash G)) & \xrightarrow{d} & C^{q+1}(L^2(\Gamma \backslash G))
\end{array}
\]

The point is that $d$ is easier to study than $\bar{\delta}$ because to study $d$ we can decompose $V$ into irreducible representations and study the action of $d$ on each part.

**References**
