EXTERIOR DIFFERENTIAL SYSTEMS AND VARIATIONS OF HODGE STRUCTURES

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Abstract. Aside from the classical case of abelian varieties and K3 surfaces, the period matrices of algebraic varieties varying in a family are subject to differential constraints; i.e., they satisfy a PDE system. We will explain two algebro-geometric consequences of the integrability conditions of this PDE system. We will also discuss a related, potentially quite interesting, conjecture.

Outline

I. Exterior differential systems (EDS)

- M is a manifold
- $A^*(M)$ is the differential graded algebra of C^∞ differential forms on M

Definitions. (i) An *EDS* is given by a graded, differential ideal

$$\mathcal{I} \subset A^*(M) \;;$$

(ii) An *integral manifold* (solution) is given by $f : X \to M$ satisfying

$$f^*(\varphi) = 0, \qquad \varphi \in \mathcal{I} \;;$$

(iii) A *Pfaffian system* is the EDS generated by sections of a sub-bundle $I \subset T^*M$.

Talk given at Durham, U.K. on May 27, 2009. Based on joint work with Jim Carlson and Mark Green and on previous work with Robert Bryant.

In this talk, all EDS's will be Pfaffian systems. Associated to I is the distribution $I^\perp = W \subset TM$. Integral manifolds satisfy

$$f_* : TX \to W \ .$$

If locally I is generated by

$$\theta^\alpha = \sum_i A_i^\alpha(y) dy^i$$

then integral manifolds are solutions of

$$\begin{cases} \theta^\alpha = 0 \\ d\theta^\alpha = 0. \end{cases}$$

This is a PDE system for $y^i(x)$ where f is locally given by $x \to y^i(x)$.

Example. $\dim M = 2n+1$ and I is a line bundle locally generated by a 1-form θ with $\theta \wedge (d\theta)^n \neq 0$. By Pfaff's theorem, locally we may choose coordinates $(x_1^1, \dots, x^n, u, u_1, \dots, u_n)$ and a generator θ so that

$$\theta = du - u_i dx^i \ .$$

Integral manifolds have dimension $\leqq n$, and those of dimension n on which $dx^1 \wedge \cdots \wedge dx^n \neq 0$ are locally 1-jet graphs

$$x \to (x, u(x), \partial_{x^i} u(x)) \ .$$

Example. Any PDE system

$$F_\lambda \left(\partial_{x^i} u^\alpha(x), u^\alpha(x), x^i \right) = 0$$

can be written as an EDS

- $M = \{(p_i^\alpha, u^\alpha, x^i) : F_\lambda(p_i^\alpha, u^\alpha, x^i) = 0\}$
- $\theta^\alpha = du^\alpha - p_i^\alpha dx^i \mid_M$.

Then the usual solutions are locally n-dimensional integral manifolds on which $dx^1 \wedge \cdots \wedge dx^n \neq 0$.

Symmetries of an EDS are diffeomorphisms that preserve $\mathfrak{I}$ as

$$\begin{cases} f : M \to M \\ f^*(\mathfrak{I}) = \mathfrak{I} \end{cases}$$

These include

$$\begin{pmatrix} \text{Point} \\ \text{transformations} \end{pmatrix} \subset \begin{pmatrix} \text{gauge} \\ \text{transformations} \end{pmatrix} \subset \begin{pmatrix} \text{contact} \\ \text{transformations} \end{pmatrix} \ .$$

The last are the customary ones used; they have the fewest invariants and the most basic. EDS's provide a geometric method for studying PDE's. The equivalence method of E. Cartan is a "quasi-algorithm" for finding the invariants.

Remark for later use: W is *bracket generating* if

$$W + [W,W] + \big[W,[W,W]\big] + \cdots = TM \ .$$

In this case, if $X_1,\ldots,X_m$ is a local framing for W the operator

$$\sum_i X_i^2$$

is hypoelliptic; it behaves like an elliptic operator but with less regularity.

Definition. The *characteristic cohomology groups* are defined by

$$H^*_{\mathcal{I}}(M) =: H^*_d(M, A^*(M)/\mathcal{I}) \ .$$

If $f : X \to M$ is an integral manifold we have

$$f^* : H^*_{\mathcal{I}}(M) \to H^*(X) \ .$$

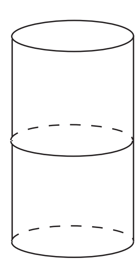
Example. For the contact system, locally

$$H^q_{\mathcal{I}}(M) = \begin{cases} \mathbb{C} & q = 0 \\ 0 & 0 < q < n \\ \dim = \infty & \text{for } q = n \end{cases}$$

The characteristic cohomology groups measure those topological properties of maps $f : X \to M$ that arise as a consequence of f satisfying a PDE system.

Example. For a determined PDE system, $H^{n-1}_{\mathcal{I}}(M) =$ "conservation laws". For $\varphi \in H^{n-1}_{\mathcal{I}}(M)$

X_t



$$\int_{X_t} f^*(\varphi) \ \text{ is independent of } t.$$

Definition. An *integral element* is $E \subset T_x M$ such that

$$\varphi(x) \mid_E = 0$$

for all $\varphi \in \mathcal{I}$.

We think of E as an infinitesimal solution to the EDS. There is a notion of *ordinary* integral elements. The *Cartan-Kähler theorem* states that in the real analytic case every ordinary integral element is tangent to a local integral manifold. We set $m_0 = \max_E \dim E$ for E ordinary. Locally, in the case where $\mathcal{I}$ is "unmixed" one has

$$H^q_{\mathcal{I}}(M) = \begin{cases} 0 & 0 < q < m_o - l \\ \dim & \infty \text{ when } q = m_0 \end{cases}$$

where l = codimension of the complex characteristic variety.

II. Period domains

Given: (H, Q), where H is a vector space over $\mathbb{Q}$, and a non-degenerate form

$$\begin{cases} Q : H \otimes H \to \mathbb{Q} \\ Q(u, v) = (-1)^n Q(v, u). \end{cases}$$

Definitions. A *Hodge structure* of weight n is given by either

(i) $\quad H_{\mathbb{C}} = \underset{p+q=n}{\oplus} H^{p,q}, \qquad H^{q,p} = \bar{H}^{p,q};$

(ii) $\quad 0 \subset F^n \subset \cdots \subset F^0 = H_{\mathbb{C}}, \qquad F^p \oplus \bar{F}^{n-p+1} \xrightarrow{\sim} \mathbb{C}.$

These are equivalent by

$$H^{p,q} = F^p \cap \bar{F}^q, \qquad F^p = \underset{p' \geqq p}{\oplus} H^{p',q'} .$$

We set $C = (\sqrt{-1})^{p-q}$ on $H^{p,q}$, $h^{p,q} = \dim H^{p,q}$ and $f^p = \sum_{p' \geqq p} h^{p',q'}$. The Hodge structure is *polarized* if the *Hodge-Riemann bilinear relations*

$$\begin{cases} Q(F^p, F^{n-p+1}) = 0 \\ Q(Cu, \bar{u}) > 0 & u \neq 0 \end{cases}$$

are satisfied.

Definitions. (i) The *period domain*

$$D = \left\{ \begin{array}{c} \text{set of polarized} \\ \text{HS's with given } h^{p,q} \end{array} \right\};$$

(ii) The *compact dual*

$$\check{D} = \left\{ \begin{array}{l} \text{set of flags with given } f^p \text{ and} \\ \text{satisfying the } 1^{\text{st}} \text{ bilinear relation} \end{array} \right\}.$$

Symmetry groups: We set

$$G = \mathrm{Aut}(H, Q) = \mathbb{Q}\text{-algebraic group}$$

and have $G_{\mathbb{R}}, G_{\mathbb{C}}$, and also $G_{\mathbb{Z}}$ if there is a lattice $H_{\mathbb{Z}}$ with $H = H_{\mathbb{Z}} \otimes \mathbb{Q}$.

Upon choice of a reference HS

$$\begin{array}{ccc} D & = & G_{\mathbb{R}}/V \\ \cap & & \\ \check{D} & = & G_{\mathbb{C}}/B \qquad V = G_{\mathbb{R}} \cap B \end{array}$$

and

$$\begin{array}{ccc} \check{D} & \subset & \prod_p \mathrm{Grass}(f^p, H_{\mathbb{C}}) \\ \cup & & \\ D & = & \text{open subset.} \end{array}$$

Thus, $\check{D}$ is a homogeneous projective variety and D is a homogeneous complex manifold.

Ex (most classical case): $D = \mathcal{H} \subset \mathbb{P}^1 = \check{D}$. To each elliptic curve = compact Riemann surface X of genus one, there is an associated period matrix = Hodge structure on $H^1(X)$ as a point in $G_{\mathbb{Z}} \backslash D = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$.

The canonical EDS on $\check{D}$. We have $T_{F^p}\,\mathrm{Grass}(f^p, H_{\mathbb{C}}) \cong \mathrm{Hom}(F^p, H_{\mathbb{C}}/F^p)$ and

$$\begin{array}{ccl} T\check{D} & \subset & \bigoplus_p \mathrm{Hom}(F^p, H_{\mathbb{C}}/F^p) \\ \cup & & \\ W & =: & T\check{D} \cap \left(\bigoplus_p \mathrm{Hom}(F^p, F^{p-1}/F^p) \right). \end{array}$$

Then the *canonical EDS on* $\check{D}$ is given by

$$I = W^{\perp} \subset T\check{D}.$$

Here, one may think of I as given by

$$dF^p \subset F^{p-1} \Leftrightarrow Q(dF^p, F^{n-p+2}).$$

It has the properties:

- I is non-trivial unless $n = 1$ (abelian varieties) or $n = 2$, $h^{2,0} = 1$ — the "classical cases" when D is a bounded symmetric domain;

- W bracket generating $\Leftrightarrow$ all $h^{p,q} \neq 0$.

Example. $n = 2$ and $h^{2,0} = 2$, $h^{1,1} = n$. Then $\dim D = 2n + 1$ and I is the contact system.

Example. When $n = 3$, $h^{3,0} = 1$, I has a local normal form. When $h^{2,1} = 1$ it is

$$\begin{cases} \theta_1 = dy' - y'' dx = 0 \\ \theta_2 = dy - y' dx = 0 \end{cases}$$

in (x, y, y', y'') space.

Aside from these cases, I is not "elementary" — it is given by honest PDE's and not just ODE's.

Let $\Gamma \subset G_{\mathbb{R}}$ be a discrete subgroup — e.g., Γ is an arithmetic group such as $G_{\mathbb{Z}}$ when $H = H_{\mathbb{Z}} \otimes \mathbb{Q}$. Then Γ acts properly discountinuously on D and

$$\Gamma \backslash D = \left\{ \begin{array}{c} \text{moduli space of} \\ \Gamma\text{-equivalence classes} \\ \text{of PHS's} \end{array} \right\}$$

is a complex analytic variety.

Definition. A *variation of Hodge structure* (VHS) is given by

$$f : S \to \Gamma \backslash D$$

where S is a complex manifold and f is a locally liftable, holomorphic mapping that is an *integral manifold of I.*

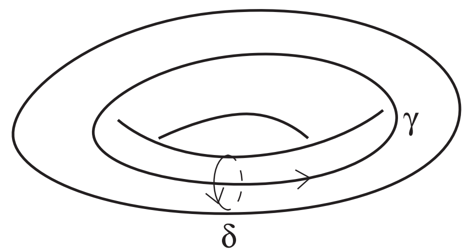
Example. Suppose given a family of smooth projective complex algebraic varieties $\{X_s\}_{s \in S}$. Then choosing a base point $s_0 \in S$ we may identify all $H^n(X_s) \cong H^n(X_{s_0})$ up to the action of monodromy. The PHS on the $H^n(X_s)$ gives a VHS where Γ includes the image of the monodromy group. In the most classical case

$$\begin{cases} X_s = \{y^2 = x(x-1)(x-s)\} \\ S = \mathbb{P}^1 \backslash \{0, 1, \infty\} \end{cases}$$

and for $\omega = dx/y = dx/\sqrt{x(x-1)(x-s)}$ the period mapping is

$$s \to \left[\int_\delta \omega, \int_\gamma \omega \right]$$

where

$X_s =$ 

Suppose now that S is quasi-projective and Γ is an arithmetic group. Then

Classical case. $\Gamma \backslash D$ is quasi-projective, defined over a number field, and $S \to \Gamma\backslash D$ is a morphism. For example, $n = 1$, $\det Q = 1$, $\Gamma = S_p(2g, \mathbb{Z})$ and then $\Gamma\backslash D = \mathcal{A}_g$ is the moduli space of principally polarized abelian varieties.

Non-classical case. $\Gamma\backslash D$ has no non-constant meromorphic functions. But the image of a VHS

$$f : S \to \Gamma\backslash D$$

is canonically a quasi-projective variety and $S \to f(S)$ is a morphism. In fact, $\mathcal{L} =: \otimes \det \mathcal{F}^p$ induces an ample line bundle on $f(S)$. Except in the classical cases, VHS is a *relative study*. The fields of definition are a pretty much unexplored territory.

III. Characteristic cohomology of period domains

For VHS's the natural global invariants come from $H^*_{\mathcal{I}}(\Gamma\backslash D)$, not $H^*(\Gamma\backslash D)$. Here, the first part to understand is that which is independent of Γ — we may think of this as *universal characteristic cohomology*. By definition this is

$$H^*_{\mathcal{I}}(D)^{G_{\mathbb{R}}} \quad =: \quad H^*_d\left((A^*(D)/\mathcal{I})^{G_{\mathbb{R}}}\right) \cong H^*(\mathcal{G}, \mathfrak{v}, \mathfrak{w})$$

where the bottom term is a Lie algebra cohomology group. The bundles $\mathcal{F}^p \to D$ and $\mathcal{H}^{p,q} \to D$ have natural metics induced by the polarization, and the Chern forms $c_i(\mathcal{F}^p)$ and $c_i(\mathcal{H}^{p,q})$ — those determine each other — are $G_{\mathbb{R}}$-invariant and over D satisfy

$$(*)\qquad \begin{cases} c_i(\mathcal{H}^{p,q}) = 0, & i > h^{p,q} \\ c(\mathcal{H}^{p,q})c(\mathcal{H}^{n-p,n-q}) = 1 \end{cases}$$

where $c(\mathcal{H}^{p,q}) = \sum_{i \geqq 0} c_i(\mathcal{H}^{p,q})$ is the total Chern form.

Denote by $I^\bullet \subset A^*(D)$ the algebraic ideal generated by I and $\bar{I}$.

Theorem. (i) $(A^*(D)/I^\bullet)^{G_{\mathbb{R}}}$ *are forms of type* (p,p). (ii) $H^*_{\mathcal{I}}(D)^{G_{\mathbb{R}}}$ *is generated by the* $c_i(\mathcal{F}^p)$, *subject to the relations* $(*)$ *and*

$$(**)\qquad c_i(\mathcal{F}^p)c_j(\mathcal{F}^{n-p}) = 0 \quad \text{if } i + j > h^{p,n-p}.$$

Remarks. $(A^*(D)/I^\bullet)^{G_{\mathbb{R}}}$ is much bigger than the part generated by the $c_i(\mathcal{F}^p)$ $(I \neq 0)$. It is only when we put in the *integrability conditions* by passing to $(A^*(D)/\mathcal{I})^{G_{\mathbb{R}}}$ that we have generation by the $c_i(\mathcal{F}^p)$ and the relation $(**)$, which is a consequence of the integrability conditions. The proof of (ii) requires rather intricate representation-theoretic considerations.

For families of algebraic varieties, $(**)$ may be formulated algebro-geometrically but there is as of now no algebro-geometric proof.

For $n = 2$, $(**)$ gives polynomial relations

$$P_k(c_1(\mathcal{H}^{2,0}), \dots, c_{h^{2,0}}(\mathcal{H}^{2,0})) = 0$$

which lead to topological conditions on the moduli spaces of surfaces of general type.

We now consider two cases

(i) Γ is co-compact and neat; i.e., has no fixed points;
(ii) Γ is arithmetic.

In case (i), $M = \Gamma \backslash D$ is a compact, complex manifold which — except in the classical cases does not even have the homotopy type of a Kähler manifold.

Recall that we denote by m_0 the maximum dimension of ordinary integral elements of $\mathcal{I}$.

Conjecture.[1] *In case* (i), *for* $m \leqq m_0$, $H^m_{\mathcal{I}}(\Gamma\backslash D)$ *has a Hodge structure of weight* m. (ii) *In case* (ii), $H^m_{\mathcal{I}}(\Gamma\backslash D)$ *has a mixed structure with weights* $m \leqq w \leqq 2m$.

In order to prove (i) it seems that two ingredients must be utilized

(a) Kähler geometry modulo $\mathcal{I}$;
(b) Hodge theory (harmonic forms, etc.) for hypoelliptic Laplacians.

Both of these would be interesting new developments at the interface of complex manifolds and EDS's. The first interesting case is when $n = 2$, $h^{2,0} = 2$, $h^{1,1} = 1$. Then M is a 3-dimensional contact manifold. Bryant has proved that

$$\dim H^1_{\mathcal{I}}(M) < \infty$$

in case (i). In fact, in this case there is a proposed sketch of a proof — not verified — of (i) in the conjecture. To give some flavor of the calculations, here are the structure equations:

- $\alpha_1, \alpha_2, \alpha_3$ is a local $(1,0)$ unitary coframe
- $I = \{\alpha_3\}$

$$\begin{cases} \boxed{\begin{aligned} d\alpha_1 &= \gamma^1_1 \wedge \alpha_1 + \gamma^2_1 \wedge \alpha_2 \\ d\alpha_2 &= \gamma^1_2 \wedge \alpha_1 + \gamma^2_2 \wedge \alpha_2 \end{aligned}} \begin{aligned} &+ \alpha_3 \wedge \bar{\alpha}_2 \\ &- \alpha_3 \wedge \bar{\alpha}_1 \end{aligned} \\ d\alpha_3 = \beta \wedge \alpha_3 + \alpha_1 \wedge \alpha_2, \qquad \beta = \gamma^1_1 + \gamma^2_2 \end{cases} \qquad \text{where } \gamma^j_i + \bar{\gamma}^i_j = 0.$$

Then

$$\mathcal{I} = \{\alpha_3, \alpha_1 \wedge \alpha_2\} + \{\bar{\alpha}_3, \bar{\alpha}_1 \wedge \bar{\alpha}_2\} .$$

Modulo $\mathcal{I}$, we have the terms in the dotted box, which look like the structure equations of a Kähler surface.

What seems to be involved here is some type of *relative Kähler geometry* where $\Delta_{\mathcal{I}} = \partial_1\partial_{\bar{1}} + \partial_2\partial_{\bar{2}}$ is hypoelliptic. The $\mathbb{Q}$-structure would presumably come from the characteristic homology groups $H_{m,\mathcal{I}}(M, \mathbb{Q})$ together with an "$\mathcal{I}$-de Rham theorem" stating that the natural pairing

$$H_{m,\mathcal{I}}(M) \otimes H^m_{\mathcal{I}}(M) \to \mathbb{C}$$

[1]We assume that all $h^{p,q} \neq 0$.

is non-degenerate for $0 \leqq m \leqq m_0$. The conjecture would imply that there is a natural map

$$\begin{pmatrix} \text{parameter spaces} \\ S \text{ for a family} \\ \text{of algebraic varieties} \end{pmatrix} \to \begin{pmatrix} \text{sub-mixed} \\ \text{Hodge structures} \\ \text{of } H^m(S) \end{pmatrix} .$$

In the non-classical cases this would be an interesting new phenomenon.

In fact, if one really wants to go out on a limb, it may be asked if, in the case when Γ is an arithmetic group,

> *Is there a number field $k \subset \mathbb{C}$, which will depend on Γ, such that the $H^m_{\mathfrak{I}}(\Gamma\backslash D)$ are defined over $\bar{k}$, meaning that there is a $\bar{k}$-vector space V^m with $V^m \otimes_{\bar{k}} \mathbb{C} \cong H^m_{\mathfrak{I}}(\Gamma\backslash D)$, together with an action of $\mathrm{Gal}(\bar{k}/k)$ on V^m?*

In other words, in the non-classical case even though $\Gamma\backslash D$ is far from being an algebraic variety defined over a number field, might the characteristic cohomology be what replaces the l-adic cohomology in the classical case?

IV. Codimension estimates of Noether-Lefschetz loci

Bottom line. For $\zeta \in H^{2p}(X, \mathbb{Q})$ the number of conditions to have

$$\zeta \in \mathrm{Hg}^p(X) = H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$$

is $h^{p-1,p+1} + \cdots + h^{0,2p}$. When X varies in a family $\{X_s\}_{s\in S}$, the N-L locus is defined as

$$S_\zeta = \{s \in S : \} \in \mathrm{Hg}^p(X_s) .$$

Then except when $p = 1$ (the classical case)

$$\boxed{\mathrm{codim}_S \;\; S_\zeta \ll h^{p-1,p+1} + \cdots + h^{0,2p} .}$$

The notation "$\ll$" is meant to suggest "much less than". This result comes about in two steps that we may summarize as follows:

- I gives $\mathrm{codim}_S(S_\zeta) \leqq h^{p-1,p+1}$
- $\mathfrak{I}$ gives $\mathrm{codim}_S(S_\zeta) \ll h^{p-1,p+1}$.

Conclusion. If the Hodge conjecture is true, then except in the classical $p = 1$ case, there are "many more" algebraic cycles than näive dimension counts suggest.

To explain a little bit of the above, in general given a manifold M and a submanifold $N \subset M$, we have for $A \subset M$

$$\operatorname{codim}_A(A \cap N) \leqq \operatorname{codim}_M(N) = \operatorname{rank}(TM/TN)$$

with equality if A is in general position relative to N.

Now let $W \subset TM$ and subject A to the differential constraint $TA \subset W\,|_A$; i.e., A is an integral manifold of $I = W^\perp \subset T^*M$. Assume that W is transverse to TN. Then

$$(*) \qquad \operatorname{codim}_A(A \cap N) \leqq \operatorname{rank}(W/W \cap TN)\ .$$

Using $dH^{p,p} \subset H^{p-1,p+1}$ this gives

$$\operatorname{codim}_S(S_\zeta) \leqq h^{p-1,p+1}\ .$$

When $p = 1$, in "most" cases (non-special divisors) equality holds here ($I = 0$ in this case). However, $(*)$ does not take the *integrability conditions* into account. In the first non-classical case $p = 2$ these are the following: Set

$$\begin{array}{rcl} T & = & T_{s_0}S \\ \cup & & \\ T_\zeta & = & \{\theta \in T_{s_0}S : \theta\zeta = 0 \text{ in } H^{p-1,p+1}\}\ . \end{array}$$

We then have

$$T_\zeta \otimes H^{4,0} \to H^{3,1}$$

and we let σ_ζ be the dimension of the image of this map. Then

$$\boxed{\operatorname{codim}_S(S_\zeta) \leqq h^{1,3} - \sigma_\zeta\ .}$$

This is the best general estimate. For example, it is an equality for smooth hypersurfaces $X \subset \mathbb{P}^5$ of degree $\geqq 6$ (to have $H^{4,0}(X) \neq 0$) containing a 2-plane.

Example. Suppose X is a Calabi-Yau fourfold, and take the local moduli space so that

$$T \to \operatorname{Hom}(H^{4,0}, H^{3,1}) \cong H^{3,1}$$

is an isomorphism (e.g. $d = 6$ above). Associated to ζ is a quadric in $\mathrm{Sym}^2 \check{T}$ defined by

$$Q_\zeta(\theta, \theta') = Q(\theta \cdot \theta'(\omega), \zeta)$$

where $\theta, \theta' \in T$ and $\omega \in H^{4,0} \cong \mathbb{C}$ is a generator. Then

If the Hodge conjecture is true and Q_ζ is non-singular, then X is defined over a number field.

Conclusion. The EDS $I \subset T^*D$, especially its integrability conditions, lead to interesting and in many cases non-classical phenomena in Hodge theory. The possible arithmetic aspects of this have yet to be explored.

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