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Final Remark. A very fine bibliography is to be found in [K].



# EXTERIOR DIFFERENTIAL SYSTEMS

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Contents



## Introduction

This is the first part of a paper where we will give an exposition of the theory of exterior differential systems, as founded by Elie Cartan and developed by Erich Kähler. We hope the paper will give an introduction to an important but still fairly inaccessible subject.

The second part will include, among others, the topics: prolongation, the Cartan-Kuranishi theorem, applications to partial differential equations and differential geometry. In writing the paper we will not be content with a treatment of an existing theory. Our principal aim is to make the formalism available for further development, such as the non-analytic case and the application to global problems, where we believe it will be useful.

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## 1. Algebraic Preliminaries

Let $V$ be a real vector space of dimension $n$ and $V^*$ its dual space. An element $x \in V$ is called a vector and an element $x' \in V^*$ a covector. $V$ and $V^*$ have a "pairing"

$$(1) \qquad (x, x') \ , \quad x \in V \ , \quad x' \in V^* \ ,$$

which is a real number and is linear in each of the arguments $x, x'$.

Over $V$ there is the exterior or Grassmann algebra, which is a graded algebra:

$$(2) \quad \Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \ldots \oplus \Lambda^n(V) \ , \quad \Lambda^0(V) = \mathbb{R} \ , \quad \Lambda^1(V) = V \ .$$

An element $\xi \in \Lambda^p(V)$ is called a multivector of degree $p$. Multiplication in $\Lambda(V)$, to be denoted by $\wedge$, is associative, distributive and satisfies the relation

$$(3) \qquad \xi \wedge \eta = (-1)^{pq} \eta \wedge \xi \ , \quad \xi \in \Lambda^p(V) \ , \quad \eta \in \Lambda^q(V) \ .$$

The multivector $\xi$ is called decomposable, if it can be written

$$(4) \qquad \xi = x_1 \wedge \ldots \wedge x_p \ , \quad x_i \in V \ .$$

$\xi \neq 0$, if and only if $x_1, \ldots, x_p$ are linearly independent, in which case $\xi$ defines the p-dimensional subspace $W$ of $V$, spanned by the



$x$'s. The decomposable multivector $\xi$ is a homogeneous coordinate of $W$, called its Grassmann coordinate; it is defined up to a non-zero factor.

In the same way there is over $V^*$ the exterior algebra

$$(5) \qquad \Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \ldots \oplus \Lambda^n(V^*) \ ,$$

$$\Lambda^0(V^*) = \mathbb{R} \ , \quad \Lambda^1(V^*) = V^* \ .$$

An element of $\Lambda^p(V^*)$ is called a form of degree $p$ or simply a p-form. If $\alpha \in \Lambda(V^*)$, we denote by $\Pi_p \alpha \in \Lambda^p(V^*)$ its component in $\Lambda^p(V^*)$.

Let $e_i$ be a base of $V$ and $e'^k$ its dual base, so that

$$(6) \qquad \langle e_i, e'^k \rangle = \delta_i^k \ , \quad 1 \leqslant i \ , \quad k \leqslant n \ .$$

Then an element $\xi \in \Lambda^p(V)$ can be written

$$(7) \qquad \xi = \frac{1}{p!} \sum a^{i_1 \ldots i_p} e_{i_1} \wedge \ldots \wedge e_{i_p}$$

and an element $\alpha \in \Lambda^p(V^*)$ as

$$(8) \qquad \alpha = \frac{1}{p!} \sum b_{i_1 \ldots i_p} e'^{i_1} \wedge \ldots \wedge e'^{i_p} \ .$$

In (7) and (8) the coefficients $a^{i_1 \ldots i_p}$ and $b_{i_1 \ldots i_p}$ are supposed to be anti-symmetric in any two of their indices, so that they are well-defined. The "pairing" of $\Lambda^p(V)$ and $\Lambda^p(V^*)$ is given by

(9) $$\langle \xi, \alpha \rangle = \frac{1}{p!} \sum a^{i_1 \cdots i_p} b_{i_1 \cdots i_p} .$$

It is independent of choice of base.

A pairing of $\Lambda(V)$ and $\Lambda(V^*)$ is then defined by distributivity and by the requirement that

(10) $$\langle \xi, \alpha \rangle = 0 , \quad \xi \in \Lambda^p(V) , \quad \alpha \in \Lambda^q(V^*) , \quad p \neq q .$$

Each of the exterior algebras $\Lambda(V)$ and $\Lambda(V^*)$ is a graded algebra. We define

(11) $$\tilde{\xi} = (-1)^p \xi , \quad \text{for} \quad \xi \in \Lambda^p(V) .$$

An endomorphism $f$ of the additive structure of $\Lambda(V)$ is called a <u>derivation</u> if it satisfies the conditions:

(1) $f\tilde{\xi} = \widetilde{f\xi}$ ,

(2) $f(\xi \wedge \eta) = f(\xi) \wedge \eta + \xi \wedge f(\eta)$ .

Such an endomorphism is called an <u>anti-derivation</u> if:

(1) $f\tilde{\xi} = -\widetilde{f\xi}$ ,

(2) $f(\xi \wedge \eta) = f(\xi) \wedge \eta + \tilde{\xi} \wedge f(\eta)$ .

It is said to be of degree $\ell$ if

$$f(\Lambda^p) \subset \Lambda^{p+\ell} .$$

Given $x \in V$ , we define the <u>exterior product</u>



$$e(x)\colon \Lambda(V) \to \Lambda(V)$$

by

(12) $$e(x)\xi = x \wedge \xi , \quad \xi \in \Lambda(V) .$$

By the relation

(13) $$\langle \xi, i(x)\alpha \rangle = \langle e(x)\xi, \alpha \rangle , \quad \xi \in \Lambda(V) , \quad \alpha \in \Lambda^*(V) ,$$

the "adjoint operator of $e(x)$" :

$$i(x)\colon \Lambda(V^*) \to \Lambda(V^*)$$

is defined. $i(x)$ is an anti-derivation of degree $-1$ ; it is called the <u>interior product</u>.

Exercise. If $\xi$ and $\alpha$ are given by (7) and (8) respectively and if

$$x = \sum x^i e_i ,$$

give the expressions of $e(x)\xi$ and $i(x)\alpha$ in terms of the basis.

We will consider ideals in $\Lambda(V^*)$ . A subring $I \subset \Lambda(V^*)$ is called an <u>ideal</u> if:

(1) $\alpha \in I$ implies that its homogeneous components $\Pi_p \alpha \in I$, $0 \leqslant p \leqslant n$.

(2) $\alpha \in I$ implies $\alpha \wedge \beta \in I$, $\forall \beta \in \Lambda(V^*)$. It follows that $\beta \wedge \alpha \in I$, $\forall \beta \in \Lambda(V^*)$.

A minimal set of generators of $I$ can be described by the following consideration: Given an ideal $I \subset \Lambda(V^*)$, all the $x \in V$ such that $i(x)I \subset I$ (i.e., $i(x)\alpha \in I$, $\forall \alpha \in I$) form a subspace $A_I \subset V$, to be called the <u>associated space</u> of $I$. Its annihilator $A_I^{\perp} \subset V^*$, defined to consist of all $x' \in V^*$ such that

$$\langle x, x' \rangle = 0 , \quad \forall x \in A_I ,$$

will be called the <u>dual associated space</u> of $I$. Then we have:

Theorem. <u>As an ideal, $I$ has a system of generators consisting of the elements of $\Lambda(A_I^{\perp})$</u>.

Proof. Let $A_I$ be spanned by $e_{r+1}, \ldots, e_n$, so that $A_I^{\perp}$ is spanned by $e'^1, \ldots, e'^r$. Let $\{G_i\}$ be a set of generators of $I$. Without loss of generality we assume that each $G_i$ (of degree $\geqslant 1$) is homogeneous. Suppose $G_1$ be linear, i.e.,

$$G_1 = \sum_i a_i e'^i , \quad 1 \leqslant i \leqslant n .$$

By hypothesis we have



$$i(e_\lambda)G_1 = a_\lambda \in I , \quad r+1 \leqslant \lambda \leqslant n .$$

Hence $a_\lambda = 0$ and

$$G_1 = a_1 e'^1 + \ldots + a_r e'^r \in A_I^{\perp} .$$

In other words, the generators of degree 1 belong to $A_I^{\perp}$.

Suppose that the generators of degree $\leqslant k$ belong to $\Lambda(A_I^{\perp})$. Let $G$ be a generator of degree $k+1$. We put

$$G' = G - e'^{r+1} \wedge (i(e_{r+1})G) .$$

The second term belongs to $I$ (since $e_{r+1} \in A_I$ and $I$ is an ideal). Moreover, it is generated by elements of degree $\leqslant k$. On the other hand, $i(e_{r+1})$ being an anti-derivation, we have

$$i(e_{r+1})G' = +e'^{r+1} \wedge (i(e_{r+1})^2 G) = 0 ,$$

This proves that $G'$ does not contain $e'^{r+1}$. Continuing this construction, we can replace $G$ by an element $\tilde{G}$ which does not contain $e'^{r+1}, \ldots, e'^n$ and hence belongs to $\Lambda(A_I^{\perp})$. Thus the induction is complete.

## 2. Cauchy characteristics

Let $M$ be a manifold of dimension $n$, with the local coordinates $x^1, \ldots, x^n$. An <u>exterior differential form</u> (later to be called simply a form) of degree $p$ has locally the expression

$$(1) \qquad \alpha = \frac{1}{p!} \sum_{I} a_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}, \quad 1 \leq i_1, \ldots, i_p \leq n,$$

where the coefficients are smooth functions and are anti-symmetric in any two of the indices. An exterior differential system is a system of equations

$$(2) \qquad \alpha_A = 0, \quad 1 \leq A \leq N,$$

where $\alpha_A$ are forms which are generally of different degrees.

An integral manifold of (2) is a submanifold $f: V \to M$ such that

$$(3) \qquad f^* \alpha_A = 0, \quad 1 \leq A \leq N.$$

Since (3) implies

$$df^* \alpha_A = f^* d\alpha_A = 0,$$

the problem remains the same, if we add to (2) the equations obtained by exterior differentiation.



More generally, the forms of $M$ generate a graded ring. A subring $I$ is called a <u>differential ideal</u>, if: (1) $\alpha \in I$ implies that each homogeneous component of $\alpha$ belongs to $I$; (2) $\alpha \in I$ implies that $\beta \wedge \alpha \in I$ for any form $\beta$. The differential ideal is called <u>closed</u> if $\alpha \in I$ implies $d\alpha \in I$. An integral manifold of $I$ is a submanifold $f: V \to M$ such that

$$(4) \qquad f^* \alpha = 0, \quad \forall \alpha \in I.$$

The fundamental problem in exterior differential systems is to study the integral manifolds of a closed differential ideal. We shall use interchangeably the terms differential ideal or differential system.

Any system of ordinary or partial differential equations can be expressed as an exterior differential system. For example, the equation of the first order

$$(5) \qquad F\left(x^i, z, \frac{\partial z}{\partial x^i}\right) = 0, \quad 1 \leq i \leq n$$

is equivalent to the exterior differential system

$$(5a) \qquad \begin{aligned} F(x^i, z, p_i) &= 0, \\ dF &= 0, \\ dz - \sum_i p_i dx^i &= 0, \\ \sum_i dx^i \wedge dp_i &= 0 \end{aligned}$$

in the $(2n+1)$-dimensional space $(x^i, z, p_i)$, and the classical integration problem for (5) is to search for n-dimensional integral manifolds on which

$$dx^1 \wedge \dots \wedge dx^n \neq 0 .$$

This example shows a disadvantage of exterior differential systems, namely, the large number of equations. We believe it cannot be avoided for a proper understanding of the equation. At the end of this section we will show how this formulation leads to the characteristics.

Perhaps the simplest exterior differential system is the one where the closed differential ideal $I$ is generated by forms of degree one. Let the generators be

$$(6) \qquad \alpha^1, \dots, \alpha^{n-r} ,$$

which we suppose to be linearly independent. The condition that $I$ is closed gives

$$(F) \qquad d\alpha^i \equiv 0 , \quad \text{mod } \alpha^1, \dots, \alpha^{n-r} , \quad 1 \leqslant i \leqslant n - r .$$

This condition (F) is called the <u>Frobenius condition</u>. A differential system

$$(6a) \qquad \alpha^1 = \dots = \alpha^{n-r} = 0$$

satisfying (F) is called <u>completely integrable</u>.



Geometrically the $\alpha$'s span at every point $x \in M$ a subspace $W_x$ of dimension $n - r$ in the cotangent space $T_x^*$ or, what is the same, a subspace $W_x^\perp$ of dimension $r$ in the tangent space $T_x$. Following Chevalley, the data $W_x^\perp \subset T_x$, $x \in M$, is called a <u>distribution</u>. Notice that the condition (F) is intrinsic, i.e., independent of local coordinates, and is also invariant under a linear change of the $\alpha$'s.

The fundamental theorem on completely integrable systems is:

Theorem 2.1 (Frobenius). <u>Let $I$ be a closed differential ideal having as generators the linearly independent forms $\alpha^1, \dots, \alpha^{n-r}$ of degree one. In a sufficiently small neighborhood there is a coordinate system $y^1, \dots, y^n$ such that $I$ is generated by $dy^{r+1}, \dots, dy^n$.</u>

Proof. We will prove the theorem by induction on $r$. Let $r = 1$. Then the subspace $W_x^\perp \subset T_x$, $x \in M$, is of dimension 1. By a well-known theorem we can choose coordinates $y^1, \dots, y^n$, such that $W_x^\perp$ is spanned by the vector $\frac{\partial}{\partial y^1}$; then $W_x$ is spanned by $dy^2, \dots, dy^n$. The latter clearly form a set of generators of $I$. Notice that in this case the condition (F) is void.

Suppose $r \geqslant 2$ and the theorem be true for $r - 1$. Let $x^i$, $1 \leqslant i \leqslant n$, be local coordinates such that

$$\alpha^1, \dots, \alpha^{n-r} , \quad dx^r$$

are linearly independent. The differential system defined by these

$n - r + 1$ forms also satisfies the condition (F). By the induction hypothesis there are coordinates $y^i$ so that

$$dy^{r+1}, \ldots, dy^n, \quad dy^r$$

are a set of generators of the corresponding differential ideal. It follows that $dx^r$ is a linear combination of these forms or that $x^r$ is a function of $y^r, \ldots, y^n$. Without loss of generality we suppose

$$\frac{\partial x^r}{\partial y^r} \neq 0 .$$

Then $\alpha^1, \ldots, \alpha^{n-r}$ differ from $dy^{r+1}, \ldots, dy^n$ by a non-singular linear transformation mod $dx^r$. We can choose as new generators of $I$

$$\alpha'^i = dy^{r+i} + p^i dx^r, \quad 1 \leq i \leq n - r,$$

and the condition (F) remains satisfied. This gives

$$d\alpha'^i = dp^i \wedge dx^r \equiv \sum_{1 \leq \lambda \leq r-1} \frac{\partial p^i}{\partial y^\lambda} dy^\lambda \wedge dx^r \equiv 0, \quad \text{mod } \alpha'^1, \ldots, \alpha'^{n-r}.$$

It follows that

$$\frac{\partial p^i}{\partial y^\lambda} = 0, \quad 1 \leq i \leq n - r, \quad 1 \leq \lambda \leq r - 1,$$

which means that $p^i$ are functions of $y^r, \ldots, y^n$. Hence in the y-coordinates we are studying a system of $n - r$ forms of degree one



involving only the coordinates $y^r, \ldots, y^n$. This reduces to the situation settled at the beginning of this proof. Hence the induction is complete.

The theorem gives a "normal form" of a completely integrable system, i.e., the system can be written locally as

$$(7) \qquad dy^{r+1} = \ldots = dy^n = 0$$

in a suitable coordinate system. The maximal integral manifolds are

$$(7a) \qquad y^{r+1} = \text{const}, \ \ldots, \ y^n = \text{const},$$

and are therefore of dimension $r$. We say that the system defines a <u>foliation</u>, of dimension $r$ and codimension $n - r$, of which the submanifolds (7a) are the <u>leaves</u>.

The condition (F) has a formulation in terms of vector fields, which is also useful. We add to the forms (6) $r$ forms $\alpha^{n-r+1}, \ldots, \alpha^n$, so that $\alpha^i$, $1 \leq i \leq n$, are linearly independent. Then we have

$$(8) \qquad d\alpha^i = \frac{1}{2} \sum_{j,k} c^i_{jk} \alpha^j \wedge \alpha^k, \quad 1 \leq i, \ j, \ k \leq n, \quad c^i_{jk} + c^i_{kj} = 0 .$$

The condition (F) can be expressed as

$$(9) \qquad c^a_{pq} = 0, \quad 1 \leq a \leq n - r, \quad n - r + 1 \leq p, q \leq n .$$

Let $f$ be a smooth function. The equation

(10) $$df = \sum_i (X_i f)\alpha^i$$

defines n operators or vector fields $X_i$ , which form a dual base to $\alpha^j$ . Exterior differentiation of (10) gives

$$\frac{1}{2}\sum \{(X_k X_j - X_j X_k)f\}\alpha^k \wedge \alpha^j + \sum (X_i f)d\alpha^i = 0 .$$

Substituting (8) into this equation, we get

(11) $$[X_k, X_j]f = (X_k X_j - X_j X_k)f = -\sum_i c^i_{kj} X_i f .$$

Equation (11) is the dual version of (8). The vectors $X_{n-r+1}, \ldots, X_n$ span at each point $x \in M$ the subspace $W_x^\perp$ of the distribution. Hence the condition (F) or (9) can be expressed as follows: Let a distribution in M be defined by the subspaces $W_x^\perp \subset T_x$ , $\dim W_x^\perp = r$ . The condition (F) says that, for any two vector fields X, Y , such that $X_x, Y_x \in W_x^\perp$ , their bracket $[X, Y]_x \in W_x^\perp$ .

Given a closed differential ideal I , we ask whether there is a local coordinate system so that I will be generated by forms involving a smaller number of the coordinates. This question is completely answered by the notion of associated space discussed in §1.

We recall that at $x \in M$ ,

(12) $$(A_I)_x = \{\xi \in T_x \mid i(\xi)I \subset I\} .$$

A vector field $\xi$ such that $\xi_x \in (A_I)_x$ is called a <u>Cauchy characteristic</u>



<u>vector field</u> of I .

Lemma 2.2. <u>If $\xi, \eta$ are Cauchy characteristic vector fields of a closed differential ideal I , so is their bracket $[\xi, \eta]$ .</u>

Let $L_\xi$ be the Lie derivative defined by $\xi$ . It is well-known that

(13) $$L_\xi = di(\xi) + i(\xi)d .$$

Since I is closed, we have $dI \subset I$ . If $\xi$ is a characteristic vector field, we have $i(\xi)I \subset I$ . By (13) it follows that $L_\xi I \subset I$ . The lemma follows from the identity

(14) $$[L_\xi, i(\eta)] \underset{\text{def}}{=} L_\xi i(\eta) - i(\eta)L_\xi = i([\xi, \eta]) ,$$

which is valid for any two vector fields $\xi, \eta$ .

To prove (14) we observe that $L_\xi$ is a derivation and $i(\eta)$ is an anti-derivation, so that $[L_\xi, i(\eta)]$ is an anti-derivation. It therefore suffices to verify (14) when the two sides act on functions f and differentials df . Clearly, when acted on f , both sides give zero. When acted on df , we have

$$\begin{aligned}[L_\xi, i(\eta)]df &= L_\xi(\eta f) - i(\eta)d(\xi f) = [\xi, \eta]f \\ &= i([\xi, \eta])df .\end{aligned}$$

This proves (14).

Theorem 2.3. Let $I$ be a closed differential ideal whose dual associated space $A_I^{\perp}$ has constant dimension $s = n - r$. Then there is a neighborhood in which there are coordinates $(x^1, \ldots, x^r; y^1, \ldots, y^s)$ such that $I$ has a set of generators that are forms in $y^1, \ldots, y^s$.

Proof: By lemma 2.2 the differential system defined by $A_I^{\perp}$ (or, what is the same, the distribution defined by $A_I$) is completely integrable. We may choose coordinates so that the foliation so defined is given by

$$(x^1, \ldots, x^r; y^1, \ldots, y^s) \to (y^1, \ldots, y^s) .$$

For $t = (t_1, \ldots, t_r)$ we set $\phi_t = \exp(t_1 \frac{\partial}{\partial x^1} + \ldots + t_r \frac{\partial}{\partial x^r})$. This gives an action of a neighborhood of the origin in $(x^1, \ldots, x^r)$-space on $M$ (a germ of $\mathbb{R}^r$-action if you like), and since $L_{\partial/\partial x^\alpha}(I) \subseteq I$ $(1 \leq \alpha \leq r)$ it follows that

$$\phi_t^*(I) \subseteq I .$$

We choose a point $p \in M$ where $\dim I(p)$ is maximal. It may be assumed that $p$ is the origin in our coordinate system, and we choose a minimal set of algebraic generators $\theta^1, \ldots, \theta^u$ for the spaces $I(0, y)$ for $|y^\mu| < \varepsilon$, $1 \leq \mu \leq s$. Thus each form $\theta^\nu$ defines a form in $\Lambda T^*_{(0,y)}(M)$



and these generate $I(0, y)$. Using $\phi_t$ we may uniquely extend the $\theta^\nu$ to forms in $\Lambda T^*_{(x,y)}(M)$ for small $x^\alpha$. In this way we obtain generators $\theta^\nu = \theta^\nu(y, dx, dy)$ for $I$ that depend on $y^\mu$, $dy^\mu$, and $dx^\alpha$ but do not depend on $x^\alpha$. We will prove that we may choose the $\theta^\nu$ so as not to depend on $dx^\alpha$.

The proof is by induction on the degree $k$ of $\theta^\nu$. When $k = 1$ we have

$$0 = i(\frac{\partial}{\partial x^\alpha})\theta^\nu$$

which gives the claim in this case. Assume that the generators of $I$ in degree $\leq k$ do not contain $dx^\alpha$ and let $\theta^\nu$ be a new generator in degree $k + 1$. We may assume that $\theta^\nu$ involves $dx^1, \ldots, dx^s$ but does not involve $dx^{s+1}, \ldots, dx^r$ (in case $s = r$ this statement is vacuous). With the additional index range $1 \leq \alpha \leq s - 1$ write

$$\theta^\nu = dx^s \wedge \eta + \psi$$

where $\eta = \eta(dx^\alpha, y^\mu, dy^\mu)$ and $\psi = \psi(dx^\alpha, y^\mu, dy^\mu)$ do not involve $dx^s$. Then

$$i(\partial/\partial x^s)\theta^\nu = \eta$$

is an element of $I$ in degree $k$. Since $\theta^\nu$ is assumed to be a new generator in degree $k + 1$ we must have $\psi \neq 0$. We then replace $\theta^\nu$ by $\theta^\nu - dx^r \wedge \eta = \psi$, and thereby eliminate $dx^s$ from $\theta^\nu$. An obvious induction then completes the proof of Theorem 2.3.

Definition 2.4. The leaves defined by the distribution $A_x$ are called the Cauchy characteristics.

We will apply this theorem to the equation (5). Equations (5a) can be written

$$
\begin{aligned}
&F(x^i, z, p_i) = 0 \\
&dz - \sum p_i dx^i = 0 , \\
&\sum_i (F_{x^i} + F_z p_i) dx^i + \sum_i F_{p_i} dp_i = 0 , \\
&\sum_i dx^i \wedge dp_i = 0 .
\end{aligned} \tag{15}
$$

The differential ideal $I$ is generated by the left-hand members of (15) and is closed.

To determine the associated space $A_I$ consider the vector

$$\xi = \sum u^i \frac{\partial}{\partial x^i} + u \frac{\partial}{\partial z} + \sum v_i \frac{\partial}{\partial p_i} \tag{16}$$

and express the condition that the interior product $i(\xi)$ keeps $I$ stable. This gives



$$
\begin{aligned}
&u - \sum p_i u^i = 0 , \\
&\sum (F_{x^i} + F_z p_i) u^i + \sum F_{p_i} v_i = 0 , \\
&\sum u^i dp_i - \sum v_i dx^i = 0 .
\end{aligned} \tag{17}
$$

Comparing the last equation of (17) with the third equation of (15), we get

$$u^i = \lambda F_{p_i} , \quad v_i = -\lambda(F_{x^i} + F_z p_i) , \tag{18a}$$

and the first equation of (17) then gives

$$u = \lambda \sum_i p_i F_{p_i} . \tag{18b}$$

The parameter $\lambda$ being arbitrary, equations (18a) and (18b) show that $\dim A_I = 1$, i.e., the characteristic vectors at each point form a one-dimensional space. The characteristic curves in the space $(x^i, z, p_i)$, or characteristic strips in the classical terminology, are the integral curves of the differential system

$$\frac{dx^i}{F_{p_i}} = -\frac{dp_i}{F_{x^i} + F_z p_i} = \frac{dz}{\sum p_i F_{p_i}} . \tag{19}$$

These are the equations of Charpit and Lagrange. To construct an integral manifold of dimension $n$ it suffices to take a "non-characteristic"* $(n-1)$-dimensional integral manifold and draw the characteristic strips through its points. Putting it in another way, an $n$-dimensional integral manifold is generated by characteristic strips.

*"Non-characteristic means transverse to the Cauchy characteristic vector field.

We wish to apply the Cauchy characteristics to prove the following global theorem:

Theorem 2.5. Consider the differential equation

$$\sum_i \left(\frac{\partial z}{\partial x^i}\right)^2 = 1 , \quad 1 \leq i \leq n . \tag{20}$$

If $z = z(x^1, \ldots, x^n)$ is a solution for all points $(x^1, \ldots, x^n) \in E^n$ ( = n-dimensional euclidean space), then $z$ is a linear function in $x^i$ , i.e.,

$$z = \sum_i a_i x^i + b , \tag{21}$$

when $a_i$, $b$ are constants satisfying $\sum a_i^2 = 1$ .

Proof. We will denote by $E^{n+1}$ the space of $(x^1, \ldots, x^n, z)$ , so that the solution can be interpreted as a graph $\Sigma$ in $E^{n+1}$ . We will also identify $E^n$ with the hyperplane $z = 0$ . Our hypothesis says that $\Sigma$ has a one-one projection to $E^n$ . For the equation (20) the denominators in the middle term of (19) are zero, so that the Cauchy characteristics satisfy

$$p_i = \text{const.} \tag{22}$$

The equations (19) can be integrated and the Cauchy characteristic curves, when projected to $E^{n+1}$ , are the straight lines



$$x^i = x_0^i + p_i t , \quad z = z_0 + t , \tag{23}$$

where $x_0^i$ , $z_0$ are constants. The graph $\Sigma$ must have the property that it is generated by the "Cauchy lines" (23), whose projections in $E^n$ form a foliation of $E^n$ .

Our theorem for $n = 2$ follows immediately. For the only foliation of $E^2$ by straight lines is given by a family of parallel lines, say

$$x^1 = x_0^1 + \cos\theta t , \quad x^2 = x_0^2 + \sin\theta t , \quad \theta = \text{const.} ,$$

and we have

$$z = \cos\theta x^1 + \sin\theta x^2 + c , \quad c = \text{const.}$$

For any $n$ we consider the level sets $\Sigma_h = \Sigma \cap \{z = h\}$ , defined by

$$z(x^1, \ldots, x^n) = h . \tag{24}$$

Because of (20), $\Sigma_h$ is a regular hypersurface. At every point of $\Sigma$ , The Cauchy line through it cuts $\Sigma_h$ orthogonally. Let

$$\nabla z = \left(\frac{\partial z}{\partial x^1}, \ldots, \frac{\partial z}{\partial x^n}\right) \in E^n . \tag{25}$$

The map $\phi\colon \sum_0 \times \mathbb{R} \to E^n$ given by

$$(26) \qquad (x, t) \mapsto (x + t\nabla z)\ ,\quad t \in \mathbb{R}\ ,\quad x = (x^1, \ldots, x^n) \in \textstyle\sum_0$$

maps each $\{x_0\} \times \mathbb{R}$ into the projection of a Cauchy line in $E^n$. Therefore it must be a diffeomorphism. It follows that $\sum_0$ is connected. We will show that $\sum_0$ must be a hyperplane in $E^n$.

To do this, we compute the Jacobian of the map $\phi$ and use the fact that it is never zero. $\sum_0$ being a regular hypersurface on $E^n$, we use a local orthonormal frame field $(x, e_1, \ldots, e_{n-1}, e_n)$, $x \in \sum_0$, where $e_n = \nabla z$ is the unit normal at $x$. On $\sum_0$ let $\omega_\alpha$ be the dual coframe to $e_\beta$, $1 \leqslant \alpha$, $\beta \leqslant n - 1$. Then we have

$$(27) \qquad \begin{aligned} dx &= \textstyle\sum \omega_\alpha \otimes e_\alpha\ , \\ de_n &= -\textstyle\sum h_{\alpha\beta}\omega_\beta \otimes e_\alpha,\quad h_{\alpha\beta} = h_{\beta\alpha}\ , \\ &\qquad 1 \leqslant \alpha\ ,\quad \beta \leqslant n-1\ , \end{aligned}$$

so that

$$(28) \qquad II = -(dx, de_n) = \textstyle\sum h_{\alpha\beta}\omega_\alpha\omega_\beta$$

is the second fundamental form of $\sum_0$. The condition $II = 0$, or $h_{\alpha\beta} = 0$, characterizes $\sum_0$ to be a hyperplane.

From (27) we have

$$(29) \qquad d(x + te_n) = \textstyle\sum(\delta_{\alpha\beta} - th_{\alpha\beta})\omega_\beta \otimes e_\alpha + dt \otimes e_n\ .$$



Hence the pull-back (under $\phi$) of the volume element of $E^n$ is, up to a non-zero factor,

$$(30) \qquad \det(\delta_{\alpha\beta} - th_{\alpha\beta})\omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge dt\ .$$

It follows that

$$(31) \qquad \det(\delta_{\alpha\beta} - th_{\alpha\beta}) \neq 0\ ,\quad t \in \mathbb{R}\ .$$

The matrix $(h_{\alpha\beta})$ being a symmetric matrix, this is possible only when all the eigenvalues of $(h_{\alpha\beta})$ are zero, i.e., $h_{\alpha\beta} = 0$. Therefore $\sum_0$ is a hyperplane and its unit normal vector $\nabla z$ is a constant. From this the conclusion (21) follows.

## 3. Pfaffian systems

Another simple exterior differential system is one which consists of one equation

$$(1) \qquad \alpha = 0 ,$$

where $\alpha$ is a form of degree $1$. This problem was studied by Pfaff in 1814-15. The corresponding closed differential ideal $I$ has the generators $\alpha, d\alpha$. The integer $r$ defined by

$$(2) \qquad \alpha \wedge (d\alpha)^r \neq 0 , \ \alpha \wedge (d\alpha)^{r+1} = 0$$

is called the rank of the equation (1). It is invariant under the change

$$(3) \qquad \alpha \to a\alpha , \quad a \neq 0 .$$

Putting it in a different way, the two-form $d\alpha$, mod $\alpha$, has an even rank $2r$.

The study of the integral manifolds of (1) is completely cleared up by the formulation of a "normal form", as given by the

Theorem 3.1. In a neighborhood suppose the equation (1) has a constant rank $r$. Then there exists a coordinate system $w^1, \ldots, w^n$, possibly in a smaller neighborhood such that the equation becomes

$$(4) \qquad dw^1 + w^2 dw^3 + \ldots + w^{2r} dw^{2r+1} = 0 .$$



Proof. For $r = 0$ condition (2) is the Frobenius condition and the theorem becomes the Frobenius Theorem.

By induction suppose that the theorem is true for $r - 1$. Let $I$ be the ideal generated by $\alpha$ and $d\alpha$. The dual associated space $A_I^\perp$ has dimension $2r + 1$. By Theorem 2.3 there are coordinates $x^1, \ldots, x^n$ such that, by multiplying by a factor if necessary, $\alpha$ is a form in $x^1, \ldots, x^{2r+1}$. There is no loss of generality in assuming $n = 2r + 1$.

Let $J$ be the closed differential ideal generated by $d\alpha$. Since $(d\alpha)^r \neq 0$, the dual associated space $A_J^\perp$ has dimension $2r$, and $A_J$ has dimension one. Hence $J$ is generated by a 2-form $\Phi$ in $2r$ variables $y^1, \ldots, y^{2r}$, and $d\alpha$ differs from it by a factor:

$$d\alpha = a\Phi , \quad a \neq 0 .$$

We have

$$(d\alpha)^r = a^r \Phi^r \neq 0 ,$$

so that

$$\Phi^r = b(y) dy^1 \wedge \ldots \wedge dy^{2r} , \quad b(y) \neq 0 .$$

The fact that $(d\alpha)^r$ is closed gives

$$da \wedge dy^1 \wedge \ldots \wedge dy^{2r} = 0 ,$$

implying that $a$ is a function of the $y$'s . If follows that $d\alpha$ is a form in $y^1, \ldots, y^{2r}$ .

Since $d\alpha$ is itself closed, there is a non-zero one-form $\beta$ in $y^1, \ldots, y^{2r}$ such that

$$d\alpha = d\beta \ .$$

Being in a 2r-dimensional space, the form $d\beta$ , mod $\beta$ , cannot have rank $2r$ , and hence must have the rank $2r - 2$ . In other words, the pfaffian equation $\beta = 0$ has rank $r - 1$ .

To the equation $\beta = 0$ we apply the induction hypothesis and write it as

$$dz^1 + z^2 dz^3 + \ldots + z^{2r-2} dz^{2r-1} = 0 \ ,$$

so that $\beta$ itself becomes

$$\beta = u(dz^1 + z^2 dz^3 + \ldots + z^{2r-2} dz^{2r-1}) \ .$$

Since

$$d(\alpha - \beta) = 0 \ ,$$

there is a function $v$ such that



$$\alpha = dv + \beta \ .$$

By an obvious change of notation we write

$$\alpha = dw^1 + w^2 dw^3 + \ldots + w^{2r} dw^{2r+1} \ .$$

Since

$$\alpha \wedge (d\alpha)^r \neq 0 \ ,$$

the functions $w^1, \ldots, w^{2r+1}$ are independent and can be extended to a full coordinate system.

From the normal form (4) we see that the general maximal integral manifolds are of dimension $r$ and are given by

(5) $$w^1 = f(w^3, w^5, \ldots, w^{2r+1}) \ ,$$

where $f$ is an arbitrary function.

Other integral manifolds are, for instance, given by

(6) $$w^1 = f(w^3, \ldots, w^{2s+1}) \ , \quad s < r$$

$$w^{2t+1} = \text{const} \ , \quad w^{2t} \ \text{arbitrary} \ ,$$

$$s + 1 \leqslant t \leqslant r \ .$$

Theorem 3.3. *Let* $\alpha$ *be a one-form. Then* $\alpha$ *has the normal form*

(10) $\alpha = y^0 dy^1 + \ldots + y^{2r} dy^{2r+1}$ , if $r + 1 = s$ ;

(11) $\alpha = dy^1 + y^2 dy^3 + \ldots + y^{2r} dy^{2r+1}$ , if $r = s$ .

*In these expressions, the* $y$'s *are independent functions and are therefore parts of a local coordinate system.*

Proof. Let $I$ be the closed differential ideal generated by $\alpha$ and $d\alpha$ . By Theorem 3.1 there are coordinates $y^1, \ldots, y^n$ in a neighborhood such that

$$\alpha = u(dy^1 + y^2 dy^3 + \ldots + y^{2r} dy^{2r+1}) .$$

A change of notation allows us to write

$$\alpha = z^0 dy^1 + z^2 dy^3 + \ldots + z^{2r} dy^{2r+1} .$$

Then

$$(d\alpha)^{r+1} = c\, dz^0 \wedge dy^1 \wedge dz^2 \wedge dy^3 \wedge \ldots \wedge dz^{2r} \wedge dy^{2r+1} , \quad c = \text{const.} \quad c \neq 0 .$$

If $s = r + 1$ , this is $\neq 0$ , and the functions $z^0, z^2, \ldots, z^{2r}$ , $y^1, y^3, \ldots, y^{2r+1}$ are independent. This proves the normal form (10).

Consider next the case $r = s$ . Then $d\alpha$ is a two-form of rank $2r$ .



By Theorem 3.2 we can write

$$\begin{aligned} d\alpha &= dw^1 \wedge dw^2 + \ldots + dw^{2r-1} \wedge dw^{2r} \\ &= d(w^1 dw^2 + \ldots + w^{2r-1} dw^{2r}) . \end{aligned}$$

Hence the form

$$\alpha - (w^1 dw^2 + \ldots + w^{2r-1} dw^{2r})$$

is closed, and is equal to $dv$ . A change of notation gives (11).

A manifold of dimension $2r + 1$ provided with a one-form $\alpha$ , defined up to a factor, such that

$$(12) \qquad \alpha \wedge (d\alpha)^r \neq 0 ,$$

is called a *contact manifold*. An example is the projectivized cotangent bundle of a manifold, whose points are the non-zero one-forms defined up to a factor. A manifold of dimension $2r$ provided with a closed two-form of maximum rank $2r$ is called a *symplectic manifold*. Both contact manifolds and symplectic manifolds play a fundamental role in theoretical mechanics. Unlike Riemannian manifolds their local properties are fairly simple.

A Pfaffian system is given by

$$\alpha^1 = \ldots = \alpha^r = 0 , \tag{13}$$

where the $\alpha$'s are one-forms and are supposed to be linearly independent. At a point $x \in M$ the $\alpha$'s span an r-dimensional subspace $W_x \subset T_x^*$, which in turn determines its annihilator $W_x^\perp \subset T_x$. Thus a Pfaffian system is geometrically a subbundle. In the above we treated the classical case $r = 1$, in which case the major invariant is the rank. In the case $r = n - 1$ the Frobenius condition is always satisfied and the system has a normal form given by Frobenius theorem. The local invariants in the general case are very complicated. For a recent work, cf. [1].



## 4. The Cartan-Kähler Theorem.

Let $M$ be a manifold of dimension $n$ and $I$ a closed differential ideal. Let $I_p$ be the set of all elements of $I$, which are homogeneous of degree $p$. A submanifold $f: V \to M$, $\dim V = p$, is called an integral manifold if

$$f^*\alpha = 0 , \quad \forall \alpha \in I . \tag{1}$$

By transitivity a submanifold of an integral manifold is an integral manifold. We suppose that the equations

$$\alpha = 0 , \quad \alpha \in I_0$$

define a submanifold of $M$. By restricting to this submanifold, we will suppose in the following that $I_0$ is empty.

If $v \in V$ and $T_vV$ is the tangent space to $V$ at $v$, then

$$\langle f_* T_v V, \alpha\rangle = \langle T_v V, f^*\alpha\rangle = 0 , \quad \forall \alpha \in I_p . \tag{2}$$

In general, if $x \in M$ and $E^p$ is a p-dimensional subspace of the tangent space $T_x$ at $x$, we say that $(x, E^p)$ is a p-dimensional integral element of $I$ if

$$\langle E^p, \alpha\rangle = 0 , \quad \forall \alpha \in I_p . \tag{3}$$

Lemma 4.1. _Let_ $E^q$ _be a q-dimensional subspace of_ $E^p$ , $q \leqslant p$ . _If_ $(x, E^p)$ _is an integral element of_ $I$ , _so is_ $(x, E^q)$ .

Proof. We choose dual bases $e_i, e^{*k}$ , $1 \leqslant i$ , $k \leqslant n$ , in $T_x$ , $T_x^*$ so that

$$\langle e_i , e^{*k} \rangle = \delta_i^k$$

and such that $e_1 \wedge \ldots \wedge e_q$ , $e_1 \wedge \ldots \wedge e_p$ define $E^q$ , $E^p$ respectively. The hypothesis can be expressed by

$$\langle e_1 \wedge \ldots \wedge e_p , \alpha \rangle = 0 , \quad \forall \alpha \in I_p .$$

Let $\beta \in I_q$ . It suffices to prove that

$$\beta \equiv 0 , \quad \text{mod } e^{*q+1} , \ldots , e^{*n} .$$

Suppose the contrary. There exists a form $\gamma$ of degree $p - q$ such that

$$\beta \wedge \gamma \equiv c e^{*1} \wedge \ldots \wedge e^{*p} , \quad c \neq 0 , \quad \text{mod } e^{*p+1} , \ldots , e^{*n} .$$

Since $I$ is an ideal, $\beta \wedge \gamma \in I_p$ . Clearly

$$\langle e_1 \wedge \ldots \wedge e_p , \beta \wedge \gamma \rangle = c \neq 0 ,$$

and we have a contradiction.



Theorem 4.2. _Let_ $I$ _be a closed differential ideal on a manifold_ $M$ . _Let_ $f\colon V \to M$ , $\dim V = p$ , _be a submanifold_. _Then_ $V$ _is an_ integral manifold if and only if $f_* T_v V$ , $v \in V$ , _are p-dimensional integral elements_. _It follows that_ $V$ _is an integral manifold if and only if_ (1) _is satisfied for all_ $\alpha \in I_p$ .

Proof. The "only if" part follows from (2). To prove the converse, take $\alpha \in I$ and suppose $\alpha$ be homogeneous. If $\deg \alpha > p$ , $f^* \alpha = 0$ for dimension reasons. If $\deg \alpha = p$ , $f^* \alpha = 0$ by hypothesis. Finally, let $\deg \alpha = q < p$ . By Lemma 4.1, $f^* \alpha = 0$ on any q-dimensional submanifold of $V$ . This is possible only when $f^* \alpha = 0$ on $V$ .

In view of this theorem the problem of finding integral manifolds of $I$ can be geometrically interpreted as "piecing together" the integral elements into a submanifold.

Given an integral element $(x, E^{p-1})$ of dimension $p - 1$ , the first step toward the construction of a p-dimensional integral manifold is to find a vector $\xi \in T_x$ such that $(x, E^{p-1} \wedge \xi)$ is an integral element of dimension $p$ . (Here we identify $E^{p-1}$ with its coordinate $(p-1)$-vector, necessarily decomposable and defined up to a non-zero factor.) The condition on $\xi$ is

(4)

$$\alpha(x, E^{p-1}, \xi) \underset{\text{def}}{=} \langle E^{p-1} \wedge \xi , \alpha \rangle = \pm \langle E^{p-1} , i(\xi) \alpha \rangle = 0 , \quad \forall \alpha \in I_p .$$

This is a system of linear homogeneous equations in $\xi$ and is satisfied whenever $\xi \in E^{p-1}$. (In classical terminology (4) is called "polarization".) The $\xi$'s satisfying (4) form a linear subspace of $T_x$, containing $E^{p-1}$. We will call it the polar space of $E^{p-1}$ and denote it by $H(E^{p-1})$. Its dimension will be denoted by $r_p + p$, $r_p \geqslant -1$; the notation is so chosen that there is no (resp. a unique) p-dimensional integral element through $E^{p-1}$ when $r_p = -1$ (resp. $= 0$). The integer $r_p$ depends on $(x, E^{p-1})$. To determine an integral element $E^p$ through $E^{p-1}$ it suffices to take in $T_x$ a subspace $Y$ of dimension $n - r_p$ through $E^{p-1}$ but otherwise in general position with $H(E^{p-1})$; $E^p$ is then the intersection of $Y$ and $H(E^{p-1})$.

Consider the Grassmann bundle $G_{p-1}(M) \to M$, whose fibre at each point $x \in M$ is the Grassmann manifold of all (p-1)-dimensional subspaces of the tangent space $T_x$. We have

(5) $$\dim G_{p-1}(M) = n + (p-1)(n-p+1) .$$

The condition for $(x, E^{p-1})$ to be an integral element is

(6) $$\beta(x, E^{p-1}) \underset{\text{def}}{=} \langle E^{p-1}, \beta\rangle = 0 , \quad \forall \beta \in I_{p-1} .$$

($E^{p-1}$, as a (p-1)-vector, is defined up to a factor, but the condition (6) is well-defined.) From now on and throughout this section we suppose all data to be real or complex analytic. It follows that the (p-1)-dimensional integral elements form an analytic subvariety of $G_{p-1}(M)$.



Definition 4.3. An integral element $(x_0, E_0^{p-1})$ is called Kähler regular or K-regular if:

1) There exist $\beta^1, \ldots, \beta^s \in I_{p-1}$ such that the subvariety $V_{p-1}(I)$ of (p-1)-dimensional integral elements on $G_{p-1}(M)$ is defined in a neighborhood of $(p_0, E_0^{p-1})$ by

(7) $$\beta^\sigma(x, E^{p-1}) = 0 , \quad 1 \leqslant \sigma \leqslant s ,$$

whose differentials $d\beta^\sigma$ are linearly independent.

2) $r_p$ is constant in a neighborhood of $(x_0, E_0^{p-1})$ on $V_{p-1}$.

It follows that $V_{p-1}$ is a manifold in a neighborhood of a K-regular integral element $(x_0, E_0^{p-1})$. Moreover, any analytic function $f$ on $G_{p-1}(M)$, which vanishes on $V_{p-1}$, can be written, in a neighborhood of $(x_0, E_0^{p-1})$, as

(8) $$f = g_1\beta^1(x, E^{p-1}) + \ldots + g_s\beta^s(x, E^{p-1}) ,$$

where the $g$'s are analytic functions.

An integral element is called K-singular if it is not K-regular. An integral manifold whose tangent spaces are K-regular integral elements is called K-regular.

Lemma 4.4. *Let* $I$ *be an analytic differential ideal,* $V_{p-1} \subset G_{p-1}(M)$ *be the variety of its* $(p-1)$*-dimensional integral elements, and* $(x_0, E_0^{p-1}) \in V_{p-1}$ *be K-regular. In a neighborhood of* $(x_0, E_0^{p-1})$ *on* $V_{p-1}$ *there are* $n - r_p - p$ *linearly independent functions in* $\xi$ *among the* $\alpha(x, E^{p-1}, \xi)$ *defined in* (4), *say* $\alpha^t(x, E^{p-1}, \xi)$, $1 \leq t \leq n - r_p - p$. *To any* $\alpha \in I_p$ *there exist analytic functions* $h_t$ *on* $G_{p-1}(M)$ *and* $g_\sigma(x, E^{p-1}, \xi)$ *on* $G_{p-1}(M) \times R^n$, *the latter being linear in* $\xi \in R^n$, *such that*

$$(9) \qquad \alpha(x, E^{p-1}, \xi) - \sum_t h_t(x, E^{p-1})\alpha^t(x, E^{p-1}, \xi)$$
$$= \sum_{1 \leq \sigma \leq s} g_\sigma(x, E^{p-1}, \xi)\beta^\sigma(x, E^{p-1}),$$

*where* $\beta^\sigma$ *are defined in Definition 4.3.*

Proof. When considered as linear functions in $\xi$, the $\alpha(x, E^{p-1}, \xi)$ and $\alpha^t(x, E^{p-1}, \xi)$, $1 \leq t \leq n - r_p - p$ have a matrix of coefficients which are analytic functions on $G_{p-1}(M)$. Restricted to $V_{p-1}$, the matrix has rank $n - r_p - p$. Hence there are $(n - r_p - p) \times (n - r_p - p)$ minors $\mu, \mu^t$ of this matrix such that

$$\mu(x, E^{p-1})\alpha(x, E^{p-1}, \xi) + \sum \mu_t(x, E^{p-1})\alpha^t(x, E^{p-1}, \xi)$$



is a linear function in $\xi$ which restricts to zero on $V_{p-1}$ and such that $\mu(x; E^{p-1})|_{V_{p-1}} \neq 0$. The lemma follows by using the remark following Definition 4.3.

The Cartan-Kähler theorem gives a construction of integral manifolds by induction on dimension. It is a local theorem and is a natural generalization of the Cauchy-Kowalewsky theorem. We now give the statements of both theorems.

Theorem 4.5. (Cauchy-Kowalewsky).

*Let* $z_1, \ldots, z_m$ *be functions in* $n$ *independent variables* $x^1, \ldots, x^n$. *Consider the system of partial differential equations*

$$(10) \qquad \frac{\partial z_i}{\partial x^n} = f_i(x^1, \ldots, x^n; z_1, \ldots, z_m; \frac{\partial z_j}{\partial x^1}, \ldots, \frac{\partial z_j}{\partial x^{n-1}}), \quad 1 \leq i, \; j \leq m$$

*where the functions* $f_i$ *are analytic in a neighborhood of the values*

$$(11) \qquad x^\alpha = x_0^\alpha, \quad z_i = z_i^0, \quad \frac{\partial z_j}{\partial x^k} = q_{jk}^0, \quad 1 \leq \alpha \leq n, \quad 1 \leq k \leq n-1.$$

*Let* $\phi_i(x^1, \ldots, x^{n-1})$ *be* $m$ *functions, which are analytic in a neighborhood of* $(x_0^1, \ldots, x_0^{n-1})$ *and satisfy the initial conditions*

$$(12) \qquad \phi_i(x_0^1, \ldots, x_0^{n-1}) = z_i^0, \quad \left(\frac{\partial \phi_i}{\partial x^k}\right)_0 = q_{ik}^0.$$

Then the system (5) has a uniquely determined system of solutions

(13) $$z_i = \phi_i(x^1, \ldots, x^n)$$

satisfying the initial conditions

(14) $$\phi_i(x^1, \ldots, x^{n-1}, x_0^n) = \phi_i(x^1, \ldots, x^{n-1}) .$$

Theorem 4.6 (Cartan-Kähler)

On an analytic manifold $M$ of dimension $n$ let $I$ be a closed differential ideal. Let $N^{p-1}$ be a K-regular $(p-1)$-dimensional integral manifold and let $(x_0, E_0^{p-1})$ be an integral element of $N^{p-1}$. Let $Y$ be an $(n-r_p)$-dimensional submanifold such that

(1) $N^{p-1} \subset Y$ ;

(2) At $x_0$ the tangent space $T_{x_0}Y$ , of dimension $n - r_p$ , contains exactly one p-dimensional integral element $(x_0, E_0^p)$ through $(x_0, E_0^{p-1})$.

Then in a neighborhood of $x_0$ there is a uniquely determined integral manifold $N^p$ tangent to $E_0^p$ at $x_0$ such that

(15) $$N^{p-1} \subset N^p \subset Y .$$

We will not give a proof of Theorem 4.4, which is classical. We proceed to prove Theorem 4.5.

We write $r = r_p$ . By changing coordinates if necessary, we suppose $Y$ to be defined by

(16) $$x^{n-r+1} = \ldots = x^n = 0 .$$



The integral manifold $N^{p-1} \subset Y$ can then be supposed to be given by the additional equations

(17) $$x^p = 0 , \quad x^i = \phi^i(x^1, \ldots, x^{p-1}) , \quad p+1 \leq i \leq n-r .$$

We take $x_0$ to be the origin and suppose $E_0^{p-1}$ and $E_0^p$ to be defined by

$$\left(\frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{p-1}}\right)_{x_0} \quad \text{and} \quad \left(\frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^p}\right)_{x_0}$$

respectively. The latter is the uniquely determined p-dimensional integral element containing $E_0^{p-1}$ and tangent to $Y$ at $x_0$ . The integral manifold $N^p \subset Y$ to be constructed can therefore be defined by

(18) $$x^i = \phi^i(x^1, \ldots, x^p) , \quad p+1 \leq i \leq n-r ,$$

satisfying the initial conditions

(19) $$\phi^i(x^1, \ldots, x^{p-1}, 0) = \phi^i(x^1, \ldots, x^{p-1}) , \quad p+1 \leq i \leq n-r .$$

The unknown functions $\phi^i(x^1, \ldots, x^p)$ are to annihilate all $\alpha \in I_p$. Let

(20) $$\alpha = \frac{1}{p!} \sum a_{i_1 \ldots i_p}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_p}, \quad 1 \leq i_1, \ldots, i_p \leq n,$$

where the coefficients are analytic functions which are anti-symmetric in their indices. Let

(21) $$E^{p-1} = \frac{1}{(p-1)!} \sum \frac{\partial(x^{i_1}, \ldots, x^{i_{p-1}})}{\partial(x^1, \ldots, x^{p-1})} \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_{p-1}}},$$

(22) $$\xi = \sum \frac{\partial x^i}{\partial x^p} \frac{\partial}{\partial x^i}.$$

Here we regard $x^1, \ldots, x^p$ as independent variables and $x^i$, $p+1 \leq i \leq n$, as their functions given by (16) and (18). Then the equations in question can be written

(23) $$\alpha(x, E^{p-1}, \xi) = \frac{1}{(p-1)!} \sum a_{i_1 \ldots i_p} \frac{\partial(x^{i_1}, \ldots, x^{i_{p-1}})}{\partial(x^1, \ldots, x^{p-1})} \frac{\partial x^{i_p}}{\partial x^p} = 0,$$

$$\alpha \in I_p.$$

Since $(x_0, E_0^{p-1})$ is a regular integral element, it has a neighborhood in $V_{p-1} \subset G_{p-1}(M)$, on which $n - r - p$ of the $\alpha(x, E^{p-1}, \xi)$ are linearly independent. Call $\alpha^t$, $1 \leq t \leq n - r - p$, the corresponding p-forms and denote their coefficients by $a^t_{i_1 \ldots i_p}$. The corresponding equations can be written



(24) $$\sum_{j=p+1}^{n-r} \sum_{i_1, \ldots, i_{p-1}} a^t_{i_1 \ldots i_{p-1} j} \frac{\partial(x^{i_1}, \ldots, x^{i_{p-1}})}{\partial(x^1, \ldots, x^{p-1})} \frac{\partial x^j}{\partial x^p} =$$

$$-\sum a^t_{i_1 \ldots i_{p-1} p} \frac{\partial(x^{i_1}, \ldots, x^{i_{p-1}})}{\partial(x^1, \ldots, x^{p-1})}.$$

This is a system of $n - r - p$ linear equations in the same number of unknowns $\partial x^j / \partial x^p$, $p + 1 \leq j \leq n - r$. At $(x_0, E_0^{p-1})$ it has by hypothesis the unique solution $(\partial x^j / \partial x^p)_0 = 0$. Hence in a neighborhood of $(x_0, E_0^{p-1})$ in $G_{p-1}(M)$ the determinant of the coefficients of the linear system is not zero. The system can be solved to give

(25) $$\frac{\partial x^j}{\partial x^p} = F^j\left(x^1, \ldots, x^{n-r}, \frac{\partial(x^{i_1}, \ldots, x^{i_{p-1}})}{\partial(x^1, \ldots, x^{p-1})}\right),$$

$$p + 1 \leq j \leq n - r, \quad 1 \leq i_1, \ldots, i_{p-1} \leq n - r.$$

This is a Cauchy-Kowalewsky system in a neighborhood of $(x_0, E_0^{p-1})$. By Theorem 4.5 there exists a uniquely determined submanifold $N^p$ defined by (18) and satisfying (15).

It remains to show that $N^p$ is an integral manifold of $I$, i.e., that it annihilates all the forms $\alpha \in I_p$. By construction it annihilates $\alpha^t$, $1 \leq t \leq n-r-p$. The crux of the matter is to show that the tangent elements $(x, E^{p-1})$ of $N^p$, with $E^{p-1}$ given by (21), are (p-1)-dimensional integral elements.

We restrict to the submanifold $N^p$ on which $x^1, \ldots, x^p$ are the coordinates. The $\beta^\sigma$ in Definition 4.3 can be written

$$(26) \qquad \beta^\sigma = \sum_{1\leqslant\ell\leqslant p} (-1)^{\ell-1} B^\sigma_\ell \, dx^1 \wedge \ldots \wedge \widehat{dx^\ell} \wedge \ldots \wedge dx^p , \qquad 1 \leqslant \sigma \leqslant s .$$

We can write

$$(27) \qquad E^{p-1} = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^{p-1}} ,$$

so that

$$(28) \qquad \beta^\sigma(x, E^{p-1}) = \frac{(-1)^{p-1}}{(p-1)!} B^\sigma_p = B^\sigma , \text{ say.}$$

Let $\alpha \in I_p$. On $N^p$ we can write

$$(29) \qquad \alpha = A dx^1 \wedge \ldots \wedge dx^p .$$

By (9) in Lemma 4.4, since the $\alpha^t$, $1 \leqslant t \leqslant n - r - p$, vanish on $N^p$, we have

$$(30) \qquad A = \sum_{1\leqslant\sigma\leqslant s} g_\sigma B^\sigma ,$$

where $g_\sigma$ are analytic functions in a neighborhood of $x_0$.



Among the forms in $I_p$ are $d\beta^\sigma$ (because $I$ is closed) and $dx^\ell \wedge \beta^\sigma$, $1 \leqslant \ell \leqslant p$ (because $I$ is an ideal). We find

$$(31) \qquad \begin{aligned} d\beta^\sigma &= \left(\frac{\partial B^\sigma_1}{\partial x^1} + \ldots + \frac{\partial B^\sigma_p}{\partial x^p}\right) dx^1 \wedge \ldots \wedge dx^p \qquad 1 \leqslant \sigma \leqslant s , \\ dx^\ell \wedge \beta^\sigma &= B^\sigma_\ell \, dx^1 \wedge \ldots \wedge dx^p \qquad 1 \leqslant \ell \leqslant p . \end{aligned}$$

By (30) their coefficients are linear combinations of $B^1, \ldots, B^\sigma$. Expressing this fact, we get a Cauchy-Kowalewsky system

$$(32) \qquad \frac{\partial B^\sigma}{\partial x^p} = \text{linear combination of } B^\tau, \frac{\partial B^\tau}{\partial x^1}, \ldots, \frac{\partial B^\tau}{\partial x^{p-1}} \qquad 1 \leqslant \sigma , \ \tau \leqslant s$$

where $B^\sigma$ satisfy the initial conditions

$$(33) \qquad B^\sigma(x^1, \ldots, x^{p-1}, 0) = 0 .$$

Hence $B^\sigma = 0$ and it follows by (30) that any $\alpha \in I_p$ restricts to zero on $N^p$. This proves Theorem 4.6.

In the proof we can define $Y$ by the equations

$$(34) \qquad x^\mu = \psi^\mu(x^1, \ldots, x^p) , \quad n-r+1 \leqslant \mu \leqslant n .$$

Hence the general solution depends on $r_p$ functions in $p$ variables.

The Cartan-Kähler theorem permits us to construct integral manifolds by induction on dimension. For applications the notion of Cartan regularity plays an important role. It is given by:

Definition 4.7. An integral element $(x_0, E_0^{p-1})$ is called Cartan regular or C-regular if:

1) It contains a C-regular integral element $(x_0, E_0^{p-2})$. (This condition is empty for $p = 1$).

2) $r_p$ is constant in a neighborhood of it on $V_{p-1}$.

If only the condition 1) is satisfied, we say that $(x_0, E_p^{p-1})$ is C-ordinary.

It follows that a C-ordinary integral element $(x_0, E_0^{p-1})$ is the end-element of a nested sequence of integral elements

$$x_0 \in E_0^1 \subset E_0^2 \subset \ldots \subset E_0^{p-2} \subset E_0^{p-1}, \tag{35}$$

of which $(x_0, E_0^k)$, $1 \leq k \leq p-2$, is C-regular. (35) is called a regular integral flag. By a successive application of the Cartan-Kähler theorem 4.6, we conclude that if $(x_0, E_0^{p-1})$ is C-ordinary there is an integral manifold $N^{p-1}$ through $x_0$ and tangent to $E_0^{p-1}$.

The relation between K-regularity and C-regularity is clarified by the following theorem and example.



Theorem 4.8. If an integral element $(x_0, E_0^{p-1})$ is C-regular, then it is also K-regular.

In the example following this theorem, we will show that the converse is not true, that is, K-regularity is more general than C-regularity.

Proof. For simplicity we will assume $I_0 = \{0\}$. Then every point of $M$ is integral and, for $p = 1$, C-regularity clearly coincides with K-regularity.

In general, our hypothesis says that we have the regular integral flag (35). Suppose $E_0^q$ be spanned by $e_1^0, \ldots, e_q^0$, $q \leq p-1$. We extend these into a frame field $e_i$, $1 \leq i \leq n$, in a neighborhood of $x_0$. It follows by continuity that in a neighborhood of $x_0$, the $E^q$ spanned by $e_1, \ldots, e_q$, $q \leq p-1$, also form a regular integral flag

$$x \in E^1 \subset \ldots \subset E^{p-1}. \tag{36}$$

Let $\omega^j$, $1 \leq j \leq n$, be one-forms which are the dual coframe, so that

$$\langle e_i, \omega^j \rangle = \delta_i^j, \quad 1 \leq i, \quad j \leq n. \tag{37}$$

An element $(x, E^{p-1})$ (not necessarily integral) near $(x_0, E_0^{p-1})$ will be spanned by the vectors

(38) $$e_q - \sum_r \ell_q^r e_r, \quad 1 \leq q \leq p-1, \quad p \leq r \leq n .$$

Then the local coordinates $x^i$ of $x$ and the $\ell_q^r$ will form a local coordinate system on $G_{p-1}(M)$ .

Let

(39) $$\alpha_q^{t_q}, \quad 1 \leq t_q \leq n - r_q - q ,$$

be the q-forms which define the polar space $H(E^{q-1})$ . Then near $(x_0, E_0^{p-1})$ the variety $V_{p-1}(I)$ of C-regular (p-1)-dimensional integral elements consists exactly of those $(x, E^{p-1})$ which annihilate the (p-1)-forms

(40) $$\alpha_1^{t_1} \wedge \omega^2 \wedge \ldots \wedge \omega^{p-1}, \quad \alpha_2^{t_2} \wedge \omega^3 \wedge \ldots \wedge \omega^{p-1}, \quad \ldots, \quad \alpha_{p-1}^{t_{p-1}},$$

The latter constitute a basis of $I_{p-1}$ . By expressing the condition that the $E^{p-1}$ spanned by the vectors in (38) annihilate these forms, we get equations which are linear in $\ell_q^r$ for each $q$ and define $V_{p-1}(I)$ as a regular submanifold in $G_{p-1}(M)$ . This proves that $(x_0, E_0^{p-1})$ is K-regular.

Remark. These equations for $V_{p-1}(I)$ are clearly independent. Their number is

$$\sum_{1 \leq i \leq p-1} (n - r_i - i) = (p-1)n - \sum_i r_i - \frac{1}{2} p(p-1).$$



Since the fiber of $G_{p-1}(M)$ is of dimension $(p-1)(n-p+1)$ , it follows that

$$\dim V_{p-1}(I) = n + \sum_{1 \leq i \leq p-1} r_i - \frac{1}{2}(p-1)(p-2) .$$

In the integral flag $E_0^{q-1} \subset E_0^q$ , if a vector spans an integral element with $E_0^q$ , it does so with $E_0^{q-1}$ . Hence $H(E_0^q) \subset H(E_0^{q-1})$ , and we have

$$r_{q+1} + q + 1 \leq r_q + q .$$

We introduce the integers

(41) $$s_q = r_q - r_{q+1} - 1 \geq 0, \quad 1 \leq q \leq p-1 .$$

Corollary 4.9. <u>Let the integral element</u> $(x_0, E_0^{p-1})$ <u>be C-regular. Near it the variety</u> $V_{p-1}(I)$ <u>of integral elements has the dimension</u>

(42) $$\dim V_{p-1}(I) = n + \sum_{1 \leq i \leq p-1} r_i - \frac{1}{2}(p-1)(p-2)$$
$$= n + s_1 + 2s_2 + \ldots + (p-2)s_{p-2} + (p-1)r_{p-1} .$$

Example. In $R^5$ consider the coframe $\omega^1$ , $\omega^2$ , $\alpha$ , $\beta^1$ , $\beta^2$ , satisfying the equations

(43) $$d\omega^1 = \alpha \wedge \beta^1, \quad d\omega^2 = \alpha \wedge \beta^2,$$
$$d\beta^1 = d\beta^2 = 0, \quad d\alpha = \beta^1 \wedge \beta^2 .$$

Let $I$ be the differential system generated by $\{\omega^1, \omega^2, d\omega^1, d\omega^2\}$. It is clearly closed. It has only one two-dimensional integral element $E^2$, i.e., $\omega^1 = \omega^2 = \alpha = 0$. Hence it is K-regular.

On the other hand, the system $J$ generated by $\{\omega^1, \omega^2, \alpha\}$ is not closed. To make it so, we should add the form $\beta^1 \wedge \beta^2$. Thus it cannot have two-dimensional integral manifolds. Hence $E^2$ is not C-ordinary.



## 5. Isometric imbedding of Riemannian manifolds; molding surfaces

This section will be concerned with the local isometric imbedding of a real analytic Riemannian manifold into an Euclidean space. The main theorem is:

Theorem 5.1. (Schläfli-Cartan) A real-analytic Riemannian manifold of dimension $n$ can be locally imbedded in an Euclidean space $E^N$ of dimension $N = n(n + 1)/2$.

Let $M$ be a Riemannian manifold of dimension $n$. In local coordinates $x^i$, suppose that the Riemannian metric is

$$ds^2 = \sum_{i,k} g_{ik} dx^i dx^k , \quad 1 \leqslant i , \ j , \ k \leqslant n , \tag{1}$$

where $g_{ik} = g_{ki}$ are analytic functions of $x^j$. The isometric imbedding problem is to find functions $y^A(x^1, \ldots, x^n)$ such that

$$ds^2 = \sum_A (dy^A)^2 , \quad 1 \leqslant A \leqslant N . \tag{2}$$

In other words, the functions $y^A$ are to satisfy the differential equations

$$\sum_A \frac{\partial y^A}{\partial x^i} \frac{\partial y^A}{\partial x^k} = g_{ik} . \tag{3}$$

The solution of the problem depends on an understanding of the geometry.

We will treat the case $n = 2$, postponing the general case to a later occasion. Over $M$ let $P$ be the principal bundle of orthonormal frames $(x, e_1, e_2)$, where $x \in M$ and $e_1, e_2$ are mutually perpendicular unit vectors at $x$. Let $\omega_1, \omega_2$ be the coframe dual to $e_1, e_2$. Then the metric on $M$ is

$$ds^2 = \omega_1^2 + \omega_2^2 . \tag{4}$$

There exists a uniquely determined one-form $\omega_{12}$ in $P$, the <u>connection form</u>, so that the structure equations

$$\begin{aligned} d\omega_1 &= \omega_{12} \wedge \omega_2 , \quad d\omega_2 = \omega_1 \wedge \omega_{12} , \\ d\omega_{12} &= -K\omega_1 \wedge \omega_2 \end{aligned} \tag{5}$$

are fulfilled. $K$ is a function on $M$ and is the <u>Gaussian curvature</u>.

Similarly, in the three-dimensional Euclidean space $E^3$ consider the space $\widetilde{P}$ of all orthonormal frames $(y, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$. It is a six-dimensional manifold and can be identified with the space of all rigid motions of $E^3$. By the equations

$$\begin{aligned} dy &= \sum \tilde{\omega}_i \tilde{e}_i , \\ d\tilde{e}_i &= \sum \tilde{\omega}_{ij} \tilde{e}_j , \qquad 1 \leqslant i, \; j, k \leqslant 3 , \\ \tilde{\omega}_{ij} &+ \tilde{\omega}_{ji} = 0 , \end{aligned} \tag{6}$$

we introduce the forms $\tilde{\omega}_i, \tilde{\omega}_{ij}$. Since



$$d(dy) = d(d\tilde{e}_i) = 0 ,$$

we have the structure equations in $E^3$ ;

$$\begin{aligned} d\tilde{\omega}_i &= \sum \tilde{\omega}_j \wedge \tilde{\omega}_{ji} , \\ d\tilde{\omega}_{ij} &= \sum \tilde{\omega}_{ik} \wedge \tilde{\omega}_{kj} . \end{aligned} \tag{7}$$

The metric in $E^3$ is

$$d\tilde{s}^2 = \tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2 . \tag{8}$$

An isometric immersion is a mapping $f: M \to E^3$ such that $f^* d\tilde{s}^2 = ds^2$. It gives rise to the following diagram of mappings:

$$\begin{array}{ccc} P & \xrightarrow{\tilde{y}} & \widetilde{P} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ M & \xrightarrow{y} & E^3 \end{array} \tag{9}$$

Here $\pi$ and $\tilde{\pi}$ are the respective projections assigning to a frame its origin, $y$ is the isometric immersion, and $\tilde{y}$ sends the orthonormal frame $(x, e_1, e_2)$ to the frame $(y(x), y_*(e_1), y_*(e_2), \tilde{e}_3)$, where $y_*$ is the mapping induced by $y$ on tangent vectors and $\tilde{e}_3 = y_*(e_1) \times y_*(e_2)$, the latter being the vector product. $\tilde{y}$ sends orthonormal frames into orthonormal frames because $y$ is an <u>isometric</u> immersion. The diagram (9) is clearly commutative. It leads to the differential system

$$\tilde{\omega}_1 - \omega_1 = \tilde{\omega}_2 - \omega_2 = \tilde{\omega}_3 = 0 \tag{10}$$

in the 9-dimensional space $P \times \widetilde{P}$. Clearly, a solution of (10), satisfying $\omega_1 \wedge \omega_2 \neq 0$, gives an isometric immersion of $M$ in $E^3$.

The system (10) is not closed. Exterior differentiation of the first two equations of (10) gives, by the use of the structure equations (5) and (7),

$$(\widetilde{\omega}_{12} - \omega_{12}) \wedge \omega_1 = (\widetilde{\omega}_{12} - \omega_{12}) \wedge \omega_2 = 0 .$$

Since $\omega_1 \wedge \omega_2 \neq 0$, this gives

$$\text{(10a)} \qquad \widetilde{\omega}_{12} - \omega_{12} = 0 .$$

Geometrically this means that the isometry preserves the connection.

It follows that the system (10) should be "prolonged" to the following:

$$\text{(11)} \qquad \widetilde{\omega}_1 - \omega_1 = \widetilde{\omega}_2 - \omega_2 = \widetilde{\omega}_3 = \widetilde{\omega}_{12} - \omega_{12} = 0 .$$

The exterior derivatives of the first two equations are now identically satisfied, as a consequence of the equations themselves, and the exterior derivatives of the last two equations give

$$\text{(11a)} \qquad \begin{aligned} \omega_1 \wedge \widetilde{\omega}_{13} + \omega_2 \wedge \widetilde{\omega}_{23} &= 0 , \\ \widetilde{\omega}_{13} \wedge \widetilde{\omega}_{23} - K\omega_1 \wedge \omega_2 &= 0 . \end{aligned}$$

The isometric imbedding problem is thus reduced to the system consisting of the equations (11), (11a), which is closed.



We seek two-dimensional integral elements $E^2$ given by

$$\text{(12)} \qquad \begin{aligned} \widetilde{\omega}_{13} &= \ell_{11}\omega_1 + \ell_{12}\omega_2 , \\ \widetilde{\omega}_{23} &= \ell_{21}\omega_1 + \ell_{22}\omega_2 , \end{aligned}$$

such that $\omega_2 = 0$ defines a regular one-dimensional integral element $E^1$ contained in it. Equations (11a) give

$$\text{(13)} \qquad \begin{aligned} \ell_{12} &= \ell_{21} \\ \ell_{11}\ell_{22} - \ell_{12}^2 &= K . \end{aligned}$$

These equations determine $\ell_{12}, \ell_{22}$, if $\ell_{11} \neq 0$. In this case the integral element $E^2$ through $E^1$ is uniquely determined. We take an integral curve, which has the integral elements $E^1$ as tangents. Through it a two-dimensional solution of the system (11), (11a) is uniquely determined. This proves Theorem 5.1 for the case $n = 2$.

More precisely, given a curve $C$ on $M$ and a curve $\widetilde{C}$ in $E^3$, we wish to construct an isometric imbedding of $M$ in $E^3$ such that $C$ goes into $\widetilde{C}$. This requires that the mapping of $C$ into $\widetilde{C}$ be an isometry, as expressed by the first equation of (11). The last equation of (11) means that $C$ and $\widetilde{C}$ have the same geodesic curvature at corresponding points. The geodesic curvature of $\widetilde{C}$ is given by $\widetilde{k} \sin\theta$, where $\widetilde{k}$ is the curvature and $\theta$ the angle which the principal normal of $\widetilde{C}$ makes with the surface normal. As a consequence we must have $\widetilde{k} \geqslant |k_g|$ at corresponding points, $k_g$ being the geodesic curvature of $C$. The equality of the geodesic curvatures at corresponding points of $C$ and $\widetilde{C}$ gives $\sin\theta = k_g/\widetilde{k}$, which in turn gives two determinations of the surface normal. Once a choice of the surface normal is made, the surface through $\tilde{C}$ and isometric to $M$ is uniquely determined, assuming $\ell_{11} \neq 0$.

The condition $\ell_{11} \neq 0$ has also a simple geometric meaning. In fact,

$$\omega_1\tilde{\omega}_{13} + \omega_2\tilde{\omega}_{23} = \ell_{11}\omega_1^2 + 2\ell_{12}\omega_1\omega_2 + \ell_{22}\omega_2^2$$

is the second fundamental form of the imbedded surface. The curve $\tilde{C}$, being defined by $\omega_2 = 0$, is an asymptotic curve if and only if $\ell_{11} = 0$. Hence the above imbedding theorem applies only to the case that $\tilde{C}$ is nonasymptotic. In fact, if $\tilde{C}$ is an asymptotic curve, its torsion is $\pm\sqrt{-K}$ (Enneper's theorem), so that $\tilde{C}$ is subject to more conditions. A corresponding isometric imbedding theorem is not contained in our general theory without a further prolongation of the differential system.

Intimately related to the problem of isometric imbedding is that of rigidity. The above discussion shows that surfaces in $E^3$ are locally not rigid, i.e., isometry does not imply congruence. It is natural to impose further conditions. In particular, we shall study the interesting problem of isometric surfaces such that the isometry preserves the lines of curvature. This study leads to the <u>molding surfaces</u> in a remarkable way.

Let $M$ be a surface in $E^3$. We shall stay away from its umbilics. Then at every point $x \in M$ we take the orthonormal frame $(x, e_1, e_2, e_3)$, where $e_1, e_2$ are along the principal directions and $e_3$ is the unit surface normal vector at $x$. This family of frames satisfies the equations



(14)
$$dx = \omega_1 e_1 + \omega_2 e_2,$$
$$de_i = \sum_j \omega_{ij} e_j,$$
$$\omega_{ij} + \omega_{ji} = 0, \qquad 1 \leq i, j, k \leq 3.$$

As in (7), we have the structure equations

(15)
$$d\omega_1 = \omega_{12} \wedge \omega_2, \qquad d\omega_2 = \omega_1 \wedge \omega_{12},$$
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}.$$

The condition that $e_1, e_2$ are along principal directions is expressed by

(16)
$$\omega_{13} = a\omega_1, \quad \omega_{23} = c\omega_2,$$

so that $K = ac$ is the Gaussian curvature. We shall suppose $K \neq 0$.

Exterior differentiation of (16) and use of (15) give

(17)
$$da \wedge \omega_1 + (a - c)\omega_{12} \wedge \omega_2 = 0,$$
$$dc \wedge \omega_2 + (a - c)\omega_{12} \wedge \omega_1 = 0.$$

Since the frame $(x, e_1, e_2, e_3)$ is completely determined at $x$, we can write

(18)
$$\omega_{12} = h\omega_1 + k\omega_2.$$

We will use $\omega_1, \omega_2$ to express the differential of any function $f$ on $M$, thus

(19) $$df = f_1\omega_1 + f_2\omega_2 ,$$

so that $f_1$, $f_2$ are "the directional derivatives" of $f$ . Using this notation, we have, from (17), (18),

(20) $$a_2 = (a - c)h ,$$
$$c_1 = (a - c)k .$$

Let $M^*$ be a surface isometric to $M$ such that the isometry preserves the lines of curvature. Using asterisks to denote the quantities pertaining to $M^*$ , we have

(21) $$\omega_1^* = \omega_1 , \quad \omega_2^* = \omega_2 , \quad \omega_3 = \omega_3^* = 0 , \quad \omega_{12}^* = \omega_{12} ,$$
$$\omega_{13}^* = ta\omega_1 , \quad \omega_{23}^* = \frac{c}{t} \wedge \omega_2 .$$

The last two equations follow from the fact that $M^*$ has the same Gaussian curvature as $M$ at corresponding points. Equation (20) gives, when applied to $M^*$ ,

(22) $$(ta)_2 = (ta - \frac{c}{t})h ,$$
$$(\frac{1}{t} c)_1 = (ta - \frac{c}{t})k .$$

Comparison of (20) and (22) gives



(23) $$t_1 = t(1 - t^2)ac^{-1}k ,$$
$$t_2 = -t^{-1}(1 - t^2)a^{-1}ch$$

or

(23a) $$\frac{tdt}{1-t^2} = t^2ac^{-1}k\omega_1 - a^{-1}ch\omega_2 .$$

From now on we suppose $t^2 \neq 1$ , discarding the trivial case that $M^*$ is congruent or symmetric to $M$ . We put

(24) $$\pi_1 = ac^{-1}k\omega_1 ,$$
$$\pi_2 = a^{-1}ch\omega_2 ,$$

so that (23a) can be written

(23b) $$\frac{tdt}{1-t^2} = t^2\pi_1 - \pi_2 .$$

Exterior differentiation of (23b) gives

(25) $$t^2(d\pi_1 - 2\pi_1 \wedge \pi_2) = d\pi_2 - 2\pi_1 \wedge \pi_2 .$$

This equation, if not satisfied identically, completely determines $t^2$ . On substituting into (23), we get conditions on the surfaces $M$ , to which there exist isometric but not congruent or symmetric surfaces preserving the lines of curvature. The latter are uniquely determined up to position in space.

The most interesting case is when the equation (25) is identically satisfied, i.e., both sides of (25) are zero. We write

(26) $$h' = a^{-1}h , \quad k' = c^{-1}k ;$$

then (24) becomes

(27) $$\pi_1 = k'\omega_{13} , \quad \pi_2 = h'\omega_{23} ,$$

and they are to satisfy the equations

(28) $$d\pi_1 = d\pi_2 = 2\pi_1 \wedge \pi_2 .$$

Substituting (27) into (28) and making use of the structure equations (15), we get

(29) $$(dh' - h'k'\omega_{13}) \wedge \omega_{23} = 0 , \quad (dk' + h'k'\omega_{23}) \wedge \omega_{13} = 0 .$$

We shall show that <u>these equations imply $hk = 0$</u>. It will be a remarkable piece of calculation; an important trick simplifying the calculation is to use $\omega_{13}$, $\omega_{23}$ in place of $\omega_1$, $\omega_2$ as the independent one-forms.

Equations (29) allow us to set



(30) $$dh' = h'k'\omega_{13} + q'\omega_{23} , \quad dk' = p'\omega_{13} - h'k'\omega_{23} .$$

On the other hand, equation (18) can be written

(31) $$\omega_{12} = h'\omega_{13} + k'\omega_{23} .$$

By (15) we have

(32) $$d\omega_{13} = \omega_{12} \wedge \omega_{23} = h'\omega_{13} \wedge \omega_{23} , \quad d\omega_{23} = -\omega_{12} \wedge \omega_{13} = k'\omega_{13} \wedge \omega_{23} .$$

Taking the exterior derivative of (31) and using (30), (32), we get

(33) $$p' - q' + 1 + h'^2 + k'^2 = 0 .$$

If $h'$ and $k'$ are considered as unknown functions, equations (30) and (33) give three relations between their directional derivatives. This primitive counting shows that the differential system is over-determined. To study our problem there is no other way but to examine the integrability conditions through differentiation of (30), (33). In this case the integrability conditions give a very simple conclusion.

We first record the formulas

(34) $$d(h'k') = (h'k'^2 + h'p')\omega_{13} + (-h'^2k' + k'q')\omega_{23} ,$$
$$\frac{1}{2} d(h'^2 + k'^2) = (h'^2k' + k'p')\omega_{13} + (-h'k'^2 + h'q')\omega_{23} ,$$

which follow from (30). Exterior differentiation of (30) gives

$$(35) \qquad (dq' + 2h'^2 k' \omega_{13}) \wedge \omega_{23} = 0 ,$$
$$(dp' + 2h' k'^2 \omega_{23}) \wedge \omega_{13} = 0 ,$$

which allow us to set

$$(36) \qquad dp' = p'' \omega_{13} - 2h' k'^2 \omega_{23} ,$$
$$dq' = -2h'^2 k' \omega_{13} + q'' \omega_{23} .$$

Differentiation of (33) then gives

$$(37) \qquad p'' = 2k' (-2h'^2 - p') ,$$
$$q'' = 2h' (-2k'^2 + q') .$$

From (33) we can set

$$(38) \qquad p' = u - \frac{1}{2} (1 + h'^2 + k'^2) ,$$
$$q' = u + \frac{1}{2} (1 + h'^2 + k'^2) ,$$

so that

$$(39) \qquad 2u = p' + q' .$$



It follows from (36)-(39) that

$$(40) \qquad du = +k' (-u + \frac{1}{2} - \frac{5}{2} h'^2 + \frac{1}{2} k'^2)\omega_{13}$$
$$+ h' (u + \frac{1}{2} + \frac{1}{2} h'^2 - \frac{5}{2} k'^2)\omega_{23} .$$

Taking the exterior derivative of this equation, we get

$$(41) \qquad h' k' = 0 \quad \text{or} \quad hk = 0 ,$$

as we have stated above.

We wish to describe these surfaces geometrically. Suppose $k = 0$. Then, by (30), (33)

$$p' = 0 , \quad q' = 1 + h'^2 .$$

It follows that the surfaces in question satisfy the equations

$$\omega_3 = 0 , \quad \omega_{13} = a\omega_1 , \quad \omega_{23} = c\omega_2 , \quad \omega_{12} = h\omega_1 ,$$
$$(42) \qquad d(\frac{h}{a}) = c(1 + \frac{h^2}{a^2})\omega_2 ,$$
$$\omega_1 \wedge da - h(a - c)\omega_1 \wedge \omega_2 = 0 .$$
$$\omega_2 \wedge dc = 0 .$$

The last three equations are obtained by exterior differentiation of the three equations before them. Hence the differential system (42) is closed.

To describe these surfaces observe that

$$\omega_1 = 0 \ , \qquad (\text{resp } \omega_2 = 0)$$

defines a family of lines of curvature, to be denoted by $\Gamma_2$ (resp $\Gamma_1$) . Along a curve of $\Gamma_2$ , we have $\omega_{12} = 0$ , so that these curves are geodesics. Writing $\omega_2 = ds$ , we have, along a curve of $\Gamma_2$ ,

$$\frac{dx}{ds} = e_2 \ , \ \frac{de_2}{ds} = ce_3 \ , \ \frac{de_3}{ds} = -ce_2 \ , \ \frac{de_1}{ds} = 0 \ .$$

Hence it is a plane curve with curvature $c$ , the plane having the normal $e_1$ . The last equation of (42) says that $dc$ is a multiple of $\omega_2$ , which means that all the curves of $\Gamma_2$ are congruent to each other.

Since

$$de_1 = \omega_{12}e_2 + \omega_{13}e_3 = (he_2 + ae_3)\omega_1 \ ,$$

the intersection of two neighboring planes of the curves of $\Gamma_2$ is a line in the direction

$$e_1 x (he_2 + ae_3) = -ae_2 + he_3 \ .$$

By (14) and (42), we have



$$d(-e_2 + \frac{h}{a} e_3) = + \frac{h}{a} \omega_{23}(-e_2 + \frac{h}{a} e_3) \ .$$

Hence this direction is fixed. It follows that the planes of the lines of curvature in $\Gamma_2$ are the tangent planes of a cylinder $Z$ .

The curves of $\Gamma_1$ , being tangent to $e_1$ , are the orthogonal trajectories of the tangent planes of $Z$ . Each line of curvature of $\Gamma_1$ is thus the locus of a point in a tangent plane of $Z$ as the latter rolls about $Z$ . The curves of $\Gamma_2$ are the orthogonal trajectories of those of $\Gamma_1$ . Each of them is therefore the position taken by a fixed curve on a tangent plane through the rolling.

The surfaces defined by (42) can be kinematically described as follows: Take a cylinder $Z$ and a curve $C$ on one of its tangent planes. The surface $M$ is the locus described by $C$ as the tangent plane rolls about $Z$ . Such a surface is called a <u>molding surface</u>. It depends on two arbitrary functions in one variable, one defining the base curve of $Z$ and the other the plane curve $C$ .

On a molding surface the equation (23b) is completely integrable and has a solution $t$ which depends on an arbitrary constant. We get in this way a non-trivial family of isometric surfaces preserving the lines of curvature (and in fact, all such families). The geometrical conclusion, reached after a lucky computation, can be stated in the following theorem:

Theorem 5.2. *In the three-dimensional Euclidean space* $E^3$ *consider two pieces of surfaces* $M, M^*$ , *such that:* (a) *their Gaussian curvature*

$\neq 0$ and they have no umbilics; (b) they are connected by an isometry $f: M \to M^*$ preserving the lines of curvature. Then $M$ and $M^*$ are in general congruent or symmetric. There are surfaces $M$, for which the corresponding $M^*$ is distinct relative to rigid motions. The molding surfaces, and only these, are such surfaces belonging to a continuous family of distinct surfaces, which are connected by isometries preserving the lines of curvature.

We observe that among the molding surfaces are the surfaces of revolution.

The computations in this case point to the unpredictable nature of the integrability conditions of an overdetermined system. In deriving such results the general theory of overdetermined systems does not seem to be very helpful.



## 6. Involution

Let $I$ be a closed differential system on a manifold $M$ and suppose that $E^p$ is a p-dimensional integral element. We ask the question: When is there an integral manifold of $I$ having $E^p$ as a tangent space? In case $E^p$ is a Cartan ordinary integral element and everything is real analytic, the existence of such an integral manifold is provided by the Cartan-Kähler theorem. However, it is clear that this is not necessary, and moreover sometimes there are additional conditions on $E^p$ that must be satisfied. In particular we wish to mention at the beginning that involution corresponds to the Cauchy initial value problem being well posed, and prolongation is nothing more than the introduction of derivatives as new variables.

In (a) we introduce the important concepts of a differential system with independence condition (cf. (6.1)) and the property of involution for such systems (cf. (6.5)).

Although the definition of involution is easy, there are some hidden subtleties involving the variety of all integral elements, and these are taken up in (b). More importantly, it is clumsy to check from the definition when a system is in involution, and so also in (b) we give a numerical criterion called Cartan's test for involution (cf. (6.29)). In a subsequent paper we will show that the homological formulation of this test using Spencer cohomology is the central technical point in the proof of the Cartan-Kuranishi prolongation theorem.