Infinitesimal variations of Hodge structure and the global Torelli problem*

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1. Introduction

The aim of this report is to introduce the infinitesimal variation of Hodge structure as a tool for studying the global Torelli problem. Although our techniques have so far borne fruit only in special cases, we are encouraged by the fact that for the first time we are able to calculate the degree of the period mapping when the Hodge-theoretic classifying space is not hermitian symmetric, as it is in the standard examples of curves, $K-3$ surfaces, and cubic threefolds. Our specific goal is therefore to prove the following:

Theorem 1. *The period mapping for cubic hypersurfaces of dimension 3m is of degree one onto its image.*

Remarks. (1) The classifying space fails to be Hermitain symmetric as soon as m exceeds unity.

(2) Our proof, unlike the Pyatetski-Shapiro-Shafarevich argument for $K-3$ surfaces, is constructive: a defining polynomial for the hypersurface is produced from the infinitesimal variation of Hodge structure.

(3) The general problem of determining when the period map is of degree one onto its image will be called the weak global Torelli problem.

In rough outline, our approach is as follows. To each variety X is associated the object $\Phi(X)$ which describes the first order variation of the Hodge structure on its kth cohomology H^k as X moves with general moduli. If the Hodge filtration satisfies $F^m \neq 0$ for $p > m$, then Φ defines a natural homomorphism

$$\phi : S^2 F^m H^k(X) \longrightarrow S^{2m-k} H^1(X, \Theta)^*$$

where S^l represents the l-th symmetric power, where $*$ denotes the dual vector space. In certain cases the variety X can be reconstructed from the kernel of the homomorphism ϕ. Now suppose that the natural map

$$p : \mathcal{M} \longrightarrow D/\Gamma$$

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from the moduli to the period space is generically finite-to-one onto its image. If it is not generically injective, we may choose a regular value y with distinct preimages in the moduli space, represented by inequivalent varieties X_1 and X_2. But p gives a local isomorphism of a neighborhood of X_i in $\mathcal{M}$ with a neighborhood of y in $p(\mathcal{M})$, and this in turn induces an isomorphism between $\Phi(X_1)$ and $\Phi(X_2)$, unique if y is not a fixed point of Γ. Thus, if there is a natural construction of X from $\Phi(X)$, then X_1 and X_2 must be isomorphic, and so the period map has degree one onto its image[1]

Remark. We argue the existence of a regular value as follows: Choose a smooth variety X which admits no automorphisms, let U be a small simply-connected open set around X in $\mathcal{M}$, let $\widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$ be the universal cover and let U be an open set in $\mathcal{M}$ which projects homeomorphically to U. Then the period map lifts to

$$\tilde{p} : \widetilde{U} \longrightarrow D;$$

and by the local Torelli theorem, its differential is of maximal rank. Consequently the only non-regular values of p on U are images of points in $\tilde{p}(\widetilde{U})$ which are fixed by some element in the arithmetic group Γ.

Now suppose that all points of $\tilde{p}(\widetilde{U})$ are fixed by an element $S \in \Gamma$. Then S commutes with the variation of Hodge structure defined by all smooth hypersurfaces of a fixed degree, hence commutes with the monodromy representation. But the monodromy representation is irreducible and contains a generalized reflection,

$$T(\gamma) = \gamma + \langle \gamma, \delta \rangle \, \delta,$$

which implies that δ, the vanishing cycle, is fixed. Since S commutes with T, we have also

$$T(\gamma) = \gamma + \langle \gamma, S\delta \rangle \, S\delta$$

which implies that δ is an eigenvector for S :

$$S\delta = \lambda\delta.$$

Schur's lemma then asserts that $S = \lambda I$, and the fact that S preserves the integral lattice implies that $\lambda = \pm 1$. Since $\pm I$ acts by the identity on all of D, the claim is demonstrated.

To see that these ideas are not totally farfetched, consider the case of curves. The infinitesimal variation determines the natural map

$$S^2 H^0(K) \longrightarrow H^1(\Theta)^* \cong H^0(2K)$$

[1] The crucial idea of iterating the differential is due to Mark Green. The general theory of infinitesimal variations of Hodge structure is the subject of a paper under preparation by two authors, Mark Green and Joe Harris.

given by cup-product. The kernel of ϕ therefore determines the linear system of quadrics passing through the canonical curve. By the Enriques-Petri theorem, the base locus of this system is the canonical model of X, provided that X is neither hyperelliptic, trigonal nor a smooth plane quartic. Since the functor $X \to \Phi(X)$ has an inverse up to isomorphism, weak global Torelli holds.

Remarks. (1) The importance of the above argument is that it proves weak global Torelli without using the Jacobian. this is an essential feature of any method which addresses the problem in higher dimension.

(2) Note that when $m = k$, the kernel of ϕ always defines a liner system of quadrics in the projective space of the canonical model X.

The example suggests that we study the kernel of ϕ in other situations. Because this study requires the detailed calculation of the cup-products in sheaf cohomology, we have restricted our investigation to hypersurfaces in projective space, where explicit formulas can be found. In this case the answer is of a somewhat different nature than for curves: given mild degree restrictions, the kernel of ϕ determines a certain homogeneous component of the Jacobian ideal of the defining equation. If this component is not zero for trivial degree reasons, we can use Macaulay's theorem to construct the vector space of first partials, and from this defining equations itself.

Now the identification of F^m with a space of homogeneous polynomials requires the use of additional data beyond the Hodge structure, namely the Poincaré residue homomorphism. Nonetheless, there are situations in which the choice of identification is irrelevant: this occurs precisely when F^m is given by residues of rational differential whose adjoint polynomial is linear. A simple calculation shows that cubics of dimension $n = 3l$ are the only hypersurfaces for which this happens: in this case $F^pH^n(X)$ is zero for $p \geq 2l$, and the residue map gives an isomorphism

$$H^0\left(\mathbb{P}_{3m+1}, \mathcal{O}(1)\right) \longrightarrow F^{2l}H^n(X)$$

To give the complete proof, then, we must (i) make precise our remarks about infinitesimal variations (ii) show how to compute cup-products on a hypersurface in terms of the adjoint polynomials which define cohomology classes through the residue map, (iii) show how to construct X from its Jacobian ideal. This will occupy us for most of the body of the paper, the real work being in (ii). Finally, we shall close with a few thoughts on the general case.

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2. Infinitesimal variations of Hodge structure

An *infinitesimal variation of Hodge structure* is a triple (H, T, Φ) consisting of (i) an S-polarized Hodge structure H of weight k, (ii) a complex vector space

T, and (iii) a graded set of homomorphisms

$$\Phi_p : T \longrightarrow \mathrm{Hom}(Gr_F^p, Gr_F^{p-1}),$$

where we have set

$$Gr_F^p = F^p/F^{p-1}.$$

The homomorphisms are subject to the compatibility relations

(iv) $\Phi_{p-1}(v)\Phi_p(w) = \Phi_{p-1}(w)\Phi_p(v)$

(v) $S(\Phi_p(v)\omega, \omega') + S(\omega, \Phi_{q+1}(v)\omega') = 0$

where $\omega \in Gr_F^p, \omega' \in Gr_F^{q+1}$, $p + q = k$. For the sake of brevity, we shall usually write Φ in place of (H, T, Φ). A morphism

$$G : \Phi \longrightarrow \Phi'$$

is given by a morphism of polarized Hodge structures,

$$G_H : H \longrightarrow H'$$

and a morphism of complex vector spaces

$$G_T : T \longrightarrow T'$$

such that

$$\Phi'(G_T\, v)(G_H\, \omega) = G_H(\Phi(v)\omega)$$

for all choices of v and ω.

We remark that an infinitesimal variation of Hodge structure is what the term suggests: a description of the derivative homomorphism of a variation of Hodge structure $B \to D$. To see that this is reasonable, observe that Φ determines a variation in a neighborhood U of B centered at zero: choose homomorphism $\tilde{\Phi}_p$ from F^p to F^{p-1} which induce Φ_p, choose coordinates t_i in U, and define a variable Hodge filtration by

$$F^p(t) = \left(1 + \sum \xi_i \tilde{\Phi}_p(\partial/\partial t_i)\right) F^p.$$

Let us now construct the associated homomorphism ϕ mentioned in the introduction. Given an r-tuple of vectors in T, we may define a homomorphism

$$\Phi^r(v) = \Phi_{p+r-1}(v_r) \circ \cdots \circ \Phi_p(v_1)$$

from Gr_F^p to Gr_F^{p+r}. The axioms of an infinitesimal variation imply that Φ^r is multilinear and symmetric in the components of v, so that we in fact obtain a homomorphism

$$\Phi^r : S^r T \longrightarrow \mathrm{Hom}(Gr_F^p, Gr_F^{p+r}).$$

Now suppose that $F^m \neq 0$ and $F^p = 0$ for $p > m$, so that Gr_F^m and Gr_F^{k-m} represent the "ends" of the Hodge filtration. Because the ends are paired perfectly by S, we may define, for each couple (ω, ω') in $F^m \times F^m$, a linear functional on $S^{2m-k}T$ by $\phi(\omega, \omega') : v \mapsto S(\Phi^{2m-k}(v)\omega, \omega')$. Applying the axioms of an infinitesimal variation once more, we find that the right hand side is symmetric and bilinear on $F^m \times F^m$. Sending (ω, ω') to the corresponding functional, we arrive at the desired homomorphism,

$$\phi : S^2F^m \longrightarrow S^{2m-k}T^*.$$

Although an infinitesimal variation defines many other natural tensors, the one just constructed is the one we shall study here. Moreover we shall only use part of the data given by Φ, namely its kernel viewed as a liner system of quadrics on S^2F^m.

3. An algebraic cup-product formula

a. Statement of the formula

The purpose of this section is to give an algebraic formula for the cup-product on a smooth projective hypersurface. To state it, we first recall how to describe rational differentials on projective space: Fix a polynomial Q, homogeneous of degree d in $n+2$ complex variables $x_0, \ldots, x_{n+1}$, and assume that the projective variety it defines is smooth:

$$X \subset \mathbb{P}_{n+1}$$

Fix a nonzero section of

$$\Omega^{n+1}_{\mathbb{P}_{n+1}}(n+2) \cong \mathcal{O}_{\mathbb{P}_{n+1}},$$

say

$$\Omega = \sum (-1)^i x_i \; dx_0 \cdots \widehat{dx_i} \cdots dx_{n+1}.$$

Then the expression

$$\Omega_A = \frac{A\Omega}{Q^{a+1}}$$

defines a rational $(n+1)$-form with X as polar locus, provided that the degree of A is chosen to make the quotient homogeneous of degree zero. The polynomial in the numerator is called an adjoint of level a, and Ω_A is said to have adjoint level a.

To each k-dimensional cohomology class on the complement of X a $(k-1)$-dimensional cohomology class is defined on X itself by the topological residue: Given a $(k-1)$-cycle γ on X, let $T(\gamma)$ be the k-cycle in $\mathbb{P}_{n+1} - X$ defined by forming the boundary of an ϵ-tubular neighborhood of γ. This construction defines a map

$$T : H_{k-1}(X) \longrightarrow H_k(\mathbb{P}_{n+1} - X)$$

whose formal adjoint, up to a factor of $2\pi\mathrm{i}$, is the topological residue. Thus, if the class on the complement is represented by a differential form, we have the explicit relation

$$\int_\gamma \operatorname{res} \alpha = \frac{1}{2\pi\mathrm{i}} \int_{T(\gamma)} \alpha.$$

The residue map satisfies the following properties[2]:

(i) res $\Gamma\Omega^{n+1}\left((n+1)X\right) = H^n\left(X,\mathbb{C}\right)_0$, where the right-hand side denotes the subspace of primitive cohomology.

(ii) res $\Gamma\Omega^{n+1}\left((a+1)X\right) = F^{n-a}H^n(X,\mathbb{C})_0$.

The compatibility between filtration by pole order and filtration by Hodge level can be strengthened further: Let Q_i denote the partial derivative of Q with respect to x_i, and let

$$J_Q = (Q_0, \ldots, Q_{n+1}),$$

the *Jacobian ideal,* be the homogeneous ideal which they generate. Then

(iii) the residue of a form Ω_A of adjoint level a has Hodge level $n-a+1$ if and only if A lies in the Jacobian ideal.

This motivates, but does not prove, the following vanishing criterion:

Theorem 2. *Let Ω_A and Ω_B be rational forms of complementary adjoint level $(a+b=n)$. Then the cup-product of their residues vanishes if and only if the ordinary product AB lies in the Jacobian ideal:*

$$\operatorname{res}\Omega_A \cdot \operatorname{res}\Omega_B = 0 \Longleftrightarrow AB \in J_Q$$

To prove this result, we first use the properties of the Hodge decomposition to show that

$$\operatorname{res}\Omega_A \cdot \operatorname{res}\Omega_B = (\operatorname{res}\Omega_A)^{n-a,a}(\operatorname{res}\Omega_B)^{n-b,b}.$$

Second, we seek explicit Čech cocycles which represent the indicated Hodge components, using

$$H^{p,q}\left(X\right) \cong H^q\left(X, \Omega^p\right).$$

[2]To keep notation reasonable, we write $\Omega^r(sX)$ for $\Omega^r_{\mathbb{P}_{n+1}}(sX)$.

Multiplication of these yields a cocycle in $H^n(X, \Omega^n)$. Third, we use the coboundary in the Poincaré residue sequence,

$$\delta : H^n(X, \Omega^n) \longrightarrow H^{n+1}\left(\mathbb{P}_{n+1}, \Omega^{n+1}\right)$$

to transfer the product to projective space. Performing all of the Čech calculations using the cover $\mathfrak{U}$ defined by

$$U_j = \{Q_j \neq 0\},$$

we obtain the $(n+1)$-cocycle

$$\eta(A, B) = c_{ab} \, \frac{AB\Omega}{Q_0 \cdots Q_{n+1}}$$

where c_{ab} is a nonzero rational constant depending only on a and b. To summarize, we claim the following:

Theorem 3. $\delta(\operatorname{res}\Omega_A \cdot \operatorname{res}\Omega_B) = \eta(A, B)$.

The vanishing criterion now follows, since it turns out that $\eta(A, B)$ cobounds precisely when AB lies in the Jacobian ideal.

Remark. It is evident that the smoothness of the X is essential, since otherwise the U_j do not cover $\mathbb{P}_{n+1}$. We shall refer to $\mathfrak{U}$ as the Jacobian cover of $\mathbb{P}_{n+1}$ relative to Q.

b. A Čechist residue formula

We shall now establish an explicit algebraic formula for the residue in Čech cohomology. To state it, we introduce the following notation: Given a vector field Z on $\mathbb{C}^{n+2}$, let $i(Z)$ denote the operation of contraction, and let $K_j = i(\delta/\delta x_j)$. Given a multi-index $J = (j_0, \ldots, j_q)$ of size q, let

$$K_J = K_{j_q} \cdots K_{j_0},$$

let

$$\Omega_J = K_J \Omega,$$

and let

$$Q_J = Q_{j_0} \cdots Q_{j_q}.$$

Proposition. *Let Ω_A be a form of adjoint level a. Then we have*

$$(\operatorname{res}\Omega_A)^{n-a,a} = c_a \left\{ \frac{A\Omega_J}{Q_J} \right\}_{|J|=q},$$

where c_a is a nonzero rational constant depending only on a.

Note that the right-hand side makes sense, since

$$\frac{A\Omega_J}{Q_J} \in \Gamma(U_J \cap X, \Omega_X^{n-a})$$

where

$$U_J = U_{j_0} \cap \cdots \cap U_{j_a}.$$

In other words, the right-hand side is an a-cocycle of the cover $\mathcal{U} \cap X$ with values in Ω_X^{n-a}.

In outline, the proof of the proposition goes as follows: The cohomology of $\mathbb{P}_{n+1} - X$ can be calculated either by the global deRham complex

(i) $\Gamma\Omega^\bullet(*X)$

of rational forms with poles of an arbitrary order on X, or by the hypercohomology of the Čech-deRham complex

(ii) $\mathcal{C}^\bullet\Omega^\bullet(\log X)$

of cocycles with a logarithmic singularity. Whereas the first complex is the natural abode of the rational forms Ω_A, it is on the second complex that the algebraic Poincaré residue is defined in hypercohomology: If

$$\omega = \left\{ \omega_I \, \frac{df_j}{f_j} \right\}$$

is a cocycle in (ii), then

$$\operatorname{res} \omega = \{ \omega_I \mid f_j = 0 \}.$$

Because both complexes compute the cohomology of the complement, there must be a Čech-deRham element with a logarithmic pole which represents Ω_A. Our problem, however, is to find an explicit representative. To do this, we imbed both resolutions in the Čech-deRham complex with arbitrary algebraic singularities,

(iii) $\mathcal{C}^\bullet\Omega^\bullet(*X)$.

Inside this object we shall construct an explicit homotopy from (i) to (ii), and from this the desired formula will follow.

We begin the real work with the definition of a partial homotopy operator: Cover $\mathbb{P}_{n+1}$ by the affine open sets

$$U_j = \{ Q_j \neq 0 \}$$

and consider the double complex

(iv) $\mathcal{C}^q(\mathcal{U}, \Omega^p(*X))$.

The cohomology of the associated single complex with respect to the total differential

$$D = d + (-1)^p \delta$$

then computes the complex cohomology of $\mathbb{P}_{n+1} - X$. Define an operator

$$H_l : \mathcal{C}^q(\mathcal{U}, \Omega^p(lX)) \longrightarrow \mathcal{C}^q(\mathcal{U}, \Omega^{p-1}((l-1)X)$$

by the formula

$$(H_l \omega)_{j_0,\ldots,j_q} = \frac{1}{1-l} \cdot \frac{Q}{Q_{j_0}} K_{j_0}\, \omega_{j_0 \ldots j_q}.$$

Lemma. *For $l \geq 2$, H satisfies the identity*

$$dH_l + H_{l+1} d \equiv 1$$

modulo the group

$$\mathcal{C}^q(\mathcal{U}, \Omega^p(l-1)X).$$

Proof. The essential point is that the exterior derivative on cochains of pole order l is represented, up to a multiplicative constant and modulo cochains of lower pole order, by multiplication against $d \log Q$:

$$d\left(\frac{\alpha}{Q^l}\right) \equiv -l \frac{dQ}{Q} \wedge \frac{\alpha}{Q^l}.$$

The identity follows from this and the commutation relation

$$K_j\, dQ + dQ K_j = Q_j.$$

The operator H gives and explicit means fo reducing the pole order of a cocycle inside (iv):

Lemma. *Let α be a Čech-deRham cochain of pole order $l \geq 2$. Then*

$$\alpha \equiv DH\alpha + HD\alpha$$

modulo cochains of pole order $l-1$. In particular, if α is a cocycle then

$$\tilde{\alpha} = (1 - DH)\alpha$$

is cohomologous to α and has pole order $l-1$.

Proof. Let α_q^p denote the component of α in $\mathcal{C}^q(U, \Omega^p(*X))$ and write

$$\begin{aligned} \alpha - DH\alpha &= \sum (\alpha_p^q - dH\alpha_q^p - (-1)^{p-1} \delta H \alpha_q^p) \\ &\equiv \sum (Hd\alpha_q^p + (-1)^p \delta H \alpha_q^p). \end{aligned}$$

Because H and δ commute up to an operator which reduces pole order, this becomes

$$\begin{aligned}\alpha - DH\alpha &\equiv \sum (Hd\alpha_q^p + (-1)^p H\delta\alpha_q^p \\ &\equiv HD\alpha\end{aligned}$$

as desired.

Remark. Let α_μ^ν denote the component of maximal Čech degree in the cocycle $\alpha : \alpha_q^p = 0$ if $q > \mu$. Then the component of maximal Čech degree for $\tilde{\alpha}$ is given by

$$\tilde{\alpha}_{\mu+1}^{\nu-1} = (-1)^\nu \delta H \alpha_\mu^\nu.$$

Let us consider the cocycle

$$\tilde{\Omega}_A = (1 - DH)a\Omega_A,$$

where Ω_A has a pole of order $a+1$. The pole order of $\tilde{\Omega}_A$ is one, and since it is D-closed,

$$d\tilde{\Omega}_A = \sum (-1)^p \delta(\Omega_A)_q^p$$

also has a pole order one. It follows that $\tilde{\Omega}_A$ has logarithmic singularities, and consequently is prepared for application of the algebraic residue map.

To calculate the residue explicitly, we calculate $(\tilde{\Omega}_A)_a$, the component of maximal Čech degree, using the following result:

Lemma.

(i) $\Omega_A \equiv (-1)^n \left\{ \frac{A\Omega_i}{Q_i Q^a} \wedge \frac{dQ}{Q} \right\}$.

(ii) *If* $I = (i_0, \ldots, i_q)$, *then*

$$H_{r+1} \left\{ \frac{A\Omega_I}{Q_I Q^r} \wedge \frac{dQ}{Q} \right\} = \frac{(-1)^{n-q+1}}{r} \left\{ \frac{A\Omega_I}{Q_I Q^r} \right\}.$$

(iii) *Under the same assumptions,*

$$\delta \left\{ \frac{A\Omega_I}{Q_I Q^r} \right\} \equiv (-1)^{q+1} \left\{ \frac{A\Omega_J}{Q_J Q^{r-1}} \wedge \frac{dQ}{Q} \right\},$$

where $J = (j_0, \ldots, j_{q+1})$.

Proof. **(i)** Let $dV = dx_0 \cdots dx_{n+1}$ be the Euclidean volume form, let

$$E = \sum x_i \frac{\partial}{\partial x_i}$$

be the Euler vector field, and observe that

$$i(E)\,dV = \Omega.$$

Take the trivial identity

$$dQ \wedge dV = 0$$

and contract with the Euler field to get

$$(\deg Q)Q\,dV - dQ \wedge \Omega = 0.$$

which we write as

$$dQ \wedge \Omega \equiv 0.$$

Contraction of this identity with $\partial/\partial x_i$ then yields

$$Q_i\Omega = dQ \wedge \Omega_i$$

or

$$\Omega \equiv (-1)^n \frac{\Omega_i}{Q_i} \wedge \frac{dQ}{Q},$$

and hence the result.

(ii) The essential point is that

$$K_{i_0}\Omega_l = 0$$

and

$$K_{i_0}\,dQ = Q_{i_0}.$$

Then

$$H_{r+1}\left\{\frac{A\Omega_I}{Q_I Q^r} \wedge \frac{dQ}{Q}\right\} = \frac{(-1)^{n-q}}{-r}\left\{\frac{Q}{Q_{i_0}} \cdot \frac{A\Omega_I}{Q_I Q^r} \wedge \frac{Q_{i_0}}{Q}\right\}$$

hence the result.

(iii) By definition of the coboundary,

$$\delta\left\{\frac{A\Omega_I}{Q_I Q^r}\right\} = \left\{\frac{A\sum(-1)^l Q_{J_l}\Omega_{J_l}}{Q_J Q^r}\right\},$$

where

$$J_l = (j_0, \ldots, \hat{j}_l, \ldots, j_{q+1}).$$

Now take the fundamental identity

$$dQ \wedge \Omega \equiv 0$$

and apply the iterated contraction operator K_J to obtain

$$(-1)^{q+1}\sum(-1)^l Q_{j_l}\Omega_{J_l} \equiv dQ\wedge\Omega_J.$$

Substitution into the formula above then yields the claimed result.

To complete the proof of the proposition, we set

$$\Omega_A(\mu) = ((1-DH)^\mu\Omega_A)_\mu^{n+1-\mu}$$

and prove inductively that

$$\Omega_A(\mu) \equiv \tilde{c}_\mu \left\{ \frac{A\Omega_J}{Q_J Q^{a-\mu}} \wedge \frac{dQ}{Q} \right\}_{|J|=\mu} \tag{I_μ}$$

where $\tilde{c}_\mu$ is a rational constant depending only on n and μ. By assertion (i) of the lemma, I_0 holds. Assume that I_μ holds, and then calculate

$$\Omega_A(\mu+1) = (-1)^{n+1-\mu}\delta H_{a-\mu+1}\Omega_A(\mu),$$

using assertions (ii) and (iii) of the lemma to obtain

$$\Omega_A(\mu+1) = \frac{(-1)^{\mu+1}}{a-\mu}\left\{ \frac{A\Omega_J}{Q_J Q^{a-\mu-1}} \wedge \frac{dQ}{Q} \right\}_{|J|=\mu+1}.$$

Thus

$$\tilde{c}_{\mu+1} = \frac{(-1)^{\mu+1}}{a-\mu}\,\tilde{c}_\mu \qquad (\mu \geq 0)$$

for $\mu > 0$ and

$$\tilde{c}_0 = (-1)^n.$$

Therefore

$$(\widetilde{\Omega}_A)_a^{n-a+1} = c_a \left\{ \frac{a\Omega_J}{Q_J} \wedge \frac{dQ}{Q} \right\}_{|J|=a} \tag{$*$}$$

with

$$c_a = \frac{(-1)^{n+a(a+1)/2}}{a!}.$$

Substitution of $(*)$ into the algebraic residue now yields the proposition.

Remark. The proof of the residue formula in Čech cohomology gives still another proof that the pole filtration is compatible with the Hodge filtration: A rational form of polar order $a+1$ unfolds in the Čech-deRham complex to a cocycle

$$\tilde{\omega}_0 + \cdots + \tilde{\omega}_a \in \bigoplus_{i\leq a} \mathcal{C}^i(\mathcal{U}, \Omega^{n+1-i}(\log X))$$

whose residue lies in

$$\bigoplus_{i\le a} \mathcal{C}^i(\mathcal{U}|_X, \Omega_X^{n-1}).$$

But the Hodge filtration on Čech-deRham cohomolog is induced by exactly this, the "filtration bête".

c. Computation of the cup-product

To complete the proof of the theorem, we must first calculate the cup-product

$$(\operatorname{res}\Omega_A)^{a,b}(\operatorname{res}\Omega_B)^{b,a},$$

where $a+b=n$, and then find its image in $H^{n+1}\left(\mathbb{P}_{n+1}, \Omega^{n+1}\right)$. To begin, let "$\wedge$" denote the natural product

$$\mathcal{C}^q(\mathcal{U}|_X, \Omega_X^p) \times \mathcal{C}^s(\mathcal{U}|_X, \Omega_X^r) \longrightarrow \mathcal{C}^{q+s}(\mathcal{U}|_X, \Omega_X^{p+r})$$

given by the "front q-face, back s-face" rule, followed by exterior multiplication of forms. Then the twisted product

$$\alpha_q^p \alpha_s^r = (-1)^{qr} \alpha_q^p \wedge \alpha_s^r$$

is skew-commutative and satisfies the Leibnitz rule [4]:

$$D(\alpha_q^p \alpha_s^r) = (D\alpha_q^p)\alpha_s^r + (-1)^{p+q}\alpha_q^p(D\alpha_s^r).$$

Because the twisted product reduces to exterior multiplication of forms, it represents the topological cup-product on the level of hypercohomology. Consequently the residue product above is represented by the cocycle

$$\{\Psi_L\} : \tilde{c}_{ab}\left\{\frac{AB\Omega_{Rs}\Omega_{sT}}{Q_R Q_s^2 Q_T}\right\} \in \mathcal{C}^n(\mathcal{U}|_X, \Omega_X^n),$$

where

$$\tilde{c}_{ab} = (-1)^{b^2} c_a c_b = \frac{(-1)^{a(a-1)/2+b(b-1)/2+b^2}}{a!\, b!}.$$

and where the multi-index L is partitioned according to

$$L = (r_0, \dots, r_{a-1}, s, t_1, \dots, t_b).$$

To calculate the coboundary of ψ in the Poincaré residue sequence, first lift to a cocycle

$$\tilde{\psi} \in \mathcal{C}^n(\mathcal{U}, \Omega^n(\log X)),$$

namely

$$\tilde{\psi}_L = \psi_L \wedge \frac{dQ}{Q},$$

and then apply the following result to simplify the numerator:

Lemma. *Let $v \in \{0, \ldots, n+1\}$ be the index complementary to L. then*

$$\Omega_{Rs} \wedge \Omega_{sT} \wedge dQ \equiv (-1)^{n+v} x_v Q_s \Omega.$$

Proof. (1) Let E denote contraction with the Euler vector field, and then write the left-hand side as

$$(K_{Rs} E\, dV) \wedge (K_{sT} E\, dV) \wedge dQ = (-1)^a (E K_{Rs}\, dV) \wedge (E K_{sT}\, dV) \wedge dQ.$$

Next, use the Euler identity

$$E\, dQ = (\deg Q) Q$$

and the fact that $E^2 = 0$ to write the right-hand side as

(i) $(-1)^a E\{(E K_{Rs}\, dV) \wedge (K_{sT}\, dV) \wedge dQ\}$, up to congruence.

(2) The ordered set $(0, \ldots, n+1)$ can be written either as

$$(0, \ldots, n+1) = RsT'vT''$$

or as

$$(0, \ldots, n+1) = R'vR''sT,$$

depending on the position of the complementary index v. We shall prove in the first case that

(ii) $(E K_{Rs}\, dV) \wedge (K_{sT}\, dV) \wedge dQ = (-1)^{b+v} x_v Q_s\, dV$;

the proof in the second is similar. To begin, note that[3]

$$K_{Rs}\, dV = dx_{T'vT''} = (-1)^{T'} dx_v\, dx_T$$

hence

(iii) $E K_{Rs}\, dV = (-1)^{T'} \{x_v\, dx_T - dx_v\, E dx_T\}$

Next, we find that

$$K_{sT}\, dV = (-1)^{RT+R+T''} dx_{Rv},$$

hence

(iv) $(K_{sT}\, dV) \wedge dQ = (-1)^{RT+R+T''} \left\{ Q_s\, dx_{Rvs} + \sum_{t \in T} Q_t\, dx_{Rvt} \right\}.$

The only nonvanishing term in the product (i) is easily seen to arise from the product of the first term in (iii) with the first term in (iv), which is

(v) $(-1)^{RT'+R+T} x_v Q_s\, dx_{TRvs}$.

[3] By convention, $(-1)^J = (-1)^q$, where $J = (j_1, \ldots, j_q)$.

Putting the dx_i in their natural order and using $v = R + T' + 1$, we obtain (ii), as desired.

(3) Substitution of (ii) into (i) then yields the asserted formula, since $E\,dV = \Omega$.

Let us now return to the calculation of $\delta\psi$: By the lemma,

$$\widetilde{\Psi}_L \equiv (-1)^{n+v}\, c_{ab} \left\{ \frac{ABx_v\Omega}{\prod\limits_{j\neq v} Q_j} \right\} \frac{1}{Q},$$

so that

$$\delta\psi \equiv (-1)^n\, \tilde{c}_{ab} \left\{ \sum \frac{ABx_vQ_v\Omega}{Q_0\cdots Q_{n+1}} \right\} \frac{1}{Q}.$$

Because of the Euler identity, this reduces to

$$\delta\psi \equiv c_{ab}\, \frac{AB\Omega}{Q_0\cdots Q_{n+1}},$$

where

$$c_{ab} \;=\; \frac{(-1)^{a(a+1)/2+b(b+1)/2+n+b^2}}{a!b!}\;\deg Q.$$

With a cocycle representative for the residue product in hand, the proof of theorem 3 is complete. To obtain the vanishing criterion of theorem 2, it therefore suffices to show that

$$\eta(A,B) \text{ cobounds } \iff AB \in J_Q.$$

To see this, consider the complexes $\mathcal{C}^\bullet_l(\mathcal{U}, \Omega^{n+1})$ where a typical cochain ϕ of dimension q has the form

$$\phi_J = \frac{R_J\Omega}{Q^l_J}.$$

Denote the cohomology groups of this complex by $H^\bullet_l(\mathbb{P}_{n+1}, \Omega_{n+1})$ and observe that

$$H^\bullet\left(\mathbb{P}_{n+1}, \Omega^{n+1}\right) = \varinjlim H^\bullet_l(\mathbb{P}_{n+1}, \Omega^{n+1}),$$

where the map from $\mathcal{C}^\bullet_l$ to $\mathcal{C}^\bullet_{l+1}$ is given by multiplication of the numerator in the cocyle against $\tilde{Q} = Q_0\cdots Q_{n+1}$. Now the residue product is represented in $\mathcal{C}^\bullet_1$, and it cobounds in this complex if and only if

$$\delta \left\{ \frac{R_j\Omega}{\prod\limits_{i\neq j} Q_i} \right\} = \eta(A,B),$$

so that

$$AB = \sum (-1)^j Q_j R_j \in J_Q.$$

Consequently

$$\eta(A, B) \text{ cobounds in } \mathcal{C}_1^\bullet \Longleftrightarrow AB \in J_Q.$$

To complete the argument, we must show that multiplication against $\tilde{Q}$ defines an injective map

$$H_l^{n+1}\left(\mathbb{P}_{n+1}, \Omega^{n+1}\right) \longrightarrow H_{l+1}^{n+1}\left(\mathbb{P}_{n+1}, \Omega^{n+1}\right),$$

hence an injective map

$$H_1^{n+1}\left(\mathbb{P}_{n+1}, \Omega^{n+1}\right) \longrightarrow H^{n+1}\left(\mathbb{P}_{n+1}, \Omega^{n+1}\right).$$

This we shall do in the next section using local duality theory.

d. The Grothendieck residue symbol and Macaulay's theorem

We begin by recalling the definition of the Grothendieck residue symbol. Let U be an open set in $\mathbb{C}^m$, let

$$f : U \longrightarrow \mathbb{C}^m$$

be holomorphic and such that $f^{-1}(0) = 0$, and let

$$\Gamma(\epsilon) = \{|f_j| = \epsilon : \ j = 1, \ldots, m\},$$

where f_j is the j-th coordinate of f. For any holomorphic function g on U, define the meromorphic differential

$$\omega_f(g) = \frac{g\, dx_1 \cdots dx_m}{f_1 \cdots f_m},$$

and set

$$\operatorname{Res}_0 \left\{ \begin{matrix} g \\ f_1 \cdots f_m \end{matrix} \right\} = \left(\frac{1}{2\pi i}\right)^m \int_{\Gamma(\epsilon)} \omega_f(g).$$

This is the Grothendieck residue symbol, which we shall usually write as $\operatorname{Res}_0(g)$. It defines a $\mathbb{C}$-linear homomorphism on the local ring $\mathcal{O}$ of holomorphic functions at the origin which annihilates the ideal

$$I = (f_1, \ldots, f_m).$$

Consequently the residue descends to a homomorphism

$$\mathrm{Res}_0 : \mathcal{O}/I \longrightarrow \mathbb{C}$$

which defines in turn a pairing

$$\mathrm{Res}_0 : (\mathcal{O}/I) \times (\mathcal{O}/I) \longrightarrow \mathbb{C}$$

by

$$(g,h) \longmapsto \mathrm{Res}_0(gh).$$

Theorem 4. *The residue pairing is perfect.*

(For proofs, see [5] or [9]).

Let us now study the residue pairing on the graded polynomial subring

$$V = \mathbb{C}[x_1, \ldots, x_m].$$

If the generators f_j are homogeneous of degree d_j, then the ideal $I \subset V$ is naturally graded, as is the quotient

$$\overline{V} = V/I.$$

If we decree that the complex numbers form an algebra of degree zero, then the residue symbol preserves the grading, up to the appropriate shift:

Lemma. *The residue symbol on $\overline{V}$ is homogeneous of degree $-\sigma$, where*

$$\sigma(I) = \sum (d_j - 1).$$

Proof. Let λ_t denote multiplication of all the variables x_j by t, let $g \in V$ be homogeneous of degree j, and observe that

$$\lambda_t^* \omega_f(g) = t^{j-\sigma} \omega_f(g).$$

The asserted homogeneity of $\omega_f(g)$ then yields the integral formula below:

$$\begin{aligned} t^{j-\sigma} \int_{\Gamma(\epsilon)} \omega_f(g) &= \int_{\Gamma(\epsilon)} \lambda_t^* \omega_f(g) \\ &= \int_{\lambda_t \Gamma(\epsilon)} \omega_f(g) = \int_{\Gamma(\epsilon)} \omega_f(g). \end{aligned}$$

Corollary. *The Grothendieck residue pairing on $\overline{V} \times \overline{V}$ satisfies the following, where the superscript indicates the natural degree:*

(i) *$\overline{V}^i$ is orthogonal to $\overline{V}^j$ unless $i + j = \sigma$.*

(ii) $\overline{V}^i$ *and* $\overline{V}^{\sigma-i}$ *are perfectly paired.*

(iii) $\overline{V}^i = 0$ *if* $i \notin [0,\sigma]$.

(iv) $\dim \overline{V}^\sigma = 1$.

As a first application of the corollary, we complete the proof of the vanishing criterion given by Theorem 2. Indeed, it suffices to prove the following:

Lemma. *The Grothendieck residue defines an isomorphism*

$$\mathrm{Res}_0 : H_l^{n+1}\left(\mathbb{P}_{n+1}, \Omega^{n+1}\right) \longrightarrow \mathbb{C}$$

which commutes with multiplication by $\tilde{Q} = Q_0 \cdots Q_{n+1}$.

Proof. Map $\mathcal{C}_l^{n+1}(\mathcal{U}, \Omega^{n+1})$ to the complex numbers by the obvious formula,

$$\frac{R\Omega}{\prod Q_j^l} \longmapsto \mathrm{Res}_0 \left\{ \frac{R}{Q_0^l \cdots Q_{n+1}^l} \right\}.$$

Because a general coboundary has the form

$$\frac{\left(\sum(-1)^j R_j Q_j^l\right)\Omega}{\prod Q_j^l} = \delta \left\{ \frac{R_j \Omega}{\prod\limits_{i\neq j} Q_i^l} \right\}$$

the set of all such can be identified with the group

$$\left\{ \frac{R\Omega}{\prod Q_j^l} \,\middle|\, R \in (Q_0^l, \ldots, Q_{n+1}^l) \right\}.$$

The residue therefore descends to a natural linear functional on H_l^{n+1}. It is an isomorphism because it factors through the identification

$$H_l^{n+1} \longrightarrow \overline{V}^\sigma,$$

where

$$\overline{V} = V/(Q_0^l, \cdots, Q_{n+1}^l).$$

It commutes with multiplication by $\tilde{Q}$ because of the obvious formula

$$\int\limits_{\Gamma_{l+1}} \frac{RQ_0 \cdots Q_{n+1}\Omega}{Q_0^{l+1} \cdots Q_{n+1}^{l+1}} = \int\limits_{\Gamma_l} \frac{R\Omega}{Q_o^l \cdots Q_{n+1}^l},$$

where

$$\Gamma_l = \{|Q_j^l| = \epsilon\} \cong \{|Q_j| = \epsilon^{1/l}\}$$

As a second application of the corollary, we shall reprove the classical Macaulay theorem. To state it, consider an ideal I in a ring V, together with a subset D of V, and define the *ideal quotient*

$$(I : D) = \{x \in V | xd \in I \text{ for all } d \in D\}.$$

The result is clearly an ideal of V containing I. let V be the graded ring $\mathbb{C}[x_i]$, and let

$$F^pV = \bigoplus_{i \geq p} V^i$$

be the filtration naturally associated to the grading. Assuming that I is generated by a regular sequence of homogeneous elements as before, we have

Macaulays Theorem.

$$(I : F^pV)^t = I + F^{\sigma-p+1}V.$$

Because the ideal quotient is itself graded, we have also the following:

Corollary. *If* $t < \sigma - p + 1$, *then*

$$(I : F^pV)^t = I^t.$$

Proof. Macaulay's theorem is equivalent to the assertion

$$\text{annihilator}(F^p\overline{V}) = F^{\sigma-p+1}\overline{V}.$$

But this follows from part (ii) of the corollary via the relation

$$\text{annihilator}(\overline{V}^i) = \bigoplus_{j \neq \sigma - i} \overline{V}^j. \qquad \text{Q.E.D.}$$

Remarks. (1) Returning to the cup-product formula, let λ be the residue of the fundamental class of $\mathbb{P}_{n+1}$, viewed in $H^{n+1}\left(\mathbb{P}_{n+1}, \Omega^{n+1}\right)$. Then we have

$$\int_X \operatorname{res}\Omega_A \operatorname{res}\Omega_B = \frac{1}{\lambda}\operatorname{Res}_0 \eta(A, B).$$

(2) One can construct an explicit "fundamental class" in $\overline{V}$, that is, an element of $\overline{V}^\sigma$ which has residue one. Such a class is in fact given by the Jacobian determinant,

$$\mathscr{J} = \det\left(\frac{\partial f_i}{\partial x_j}\right).$$

To see this, write

$$\omega_f(\mathscr{J}) = \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m},$$

and observe that

$$\int\limits_{\Gamma} \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m} = \int\limits_{f(\Gamma)} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_m}{x_m}.$$

Now $f(\Gamma)$ is an integer multiple of the fundamental class of

$$(S^1)^m \cong \{|x_i| = \epsilon\},$$

so that the integral on the right is $(2\pi \mathrm{i})^m$ times a positive integer. To evaluate the integer (which is just the multiplicity of the ideal I), deform $(f_1, \ldots, f_m)$ in the set of regular sequences to $(x_1^{d_1}, \ldots, x_m^{d_m})$. Continuity of the integral then yields

$$\mathrm{Res}_0\, \omega_f(\mathscr{J}) = d_1 \cdots d_m,$$

so that

$$\frac{\mathscr{J}}{d_1 \cdots d_m}$$

represents the fundamental class.

4. Proof of the main theorem

a. Computation of the kernel of ϕ

We now return to the proof of the Torelli theorem, the first step of which is to calculate the kernel of

$$\phi : S^2 F^m \longrightarrow S^{2m-n} T^*$$

for a fixed hypersurface X in $\mathbb{P}_{n+1}$, where $T = H^1(X, \Theta)$. In this case the homomorphism

$$H^1\,(X, \Theta) \longrightarrow \mathrm{Hom}(F^p/F^{p+1}, F^{p-1}/F^p)$$

is induced by the Gauss-Manin connection, which we can compute as follows: Let

$$Q + tR$$

be a pencil of hypersurfaces X_t with $X_0 = X$, so that differentiation with respect to t represents a tangent vector in $H^1\,(X, \Theta)$, which we shall denote also by $\partial/\partial R$. Fix a rational form Ω_A on $\mathbb{P}_{n+1} - X$, and let ω_A be its Poincaré residue. To extend ω_A to a family of cohomology classes on $\{X_t\}$, it is enough to extend Ω_A to a family of rational forms on $\{\mathbb{P}_{n+1} - X_t\}$, so we set

$$\Omega_A(t) = \frac{A\Omega}{(Q+tR)^{a+1}}$$

$$\omega_A(t) = \operatorname{res}\Omega_A(t).$$

formal differentiation of the rational forms gives

$$\text{(i)} \qquad \frac{\partial}{\partial t}\,\Omega_A(t) = -(a+1)\Omega_{RA}(t),$$

and application of the Poincaré residue yields

$$\text{(ii)} \qquad \frac{\partial}{\partial t}\,\omega_A(t) = -(a+1)\omega_{RA}(t),$$

which we may write symbolically as

$$\frac{\partial}{\partial R}\,\omega_A = -(a+1)\omega_{RA}.$$

The passage from (i) to (ii) is justified by passage of differentiation under the integral sign in the residue formula. Because the result of differentiation on the level of rational forms is independent of the extension of Ω_A to $\Omega_A(t)$, modulo forms of lower pole order, the result of differentiation on the level of residues is well-defined up to cohomology classes of higher Hodge level. In conclusion, we find that

$$\Phi\left(\frac{\partial}{\partial R}\right)\omega_A = c\,\omega_{RA}$$

for a suitable integral constant c, so that iteration yields

$$\Phi^s\left(\frac{\partial}{\partial R_1}\otimes\cdots\otimes\frac{\partial}{\partial R_s}\right)\omega_A = c\,\omega_{R_1\cdots R_s A}.$$

We shall now compute the linear functional ϕ in S^2F^m defined by

$$\phi(w,w') : v \longmapsto S(\Phi^{2m-n}(v)\omega,\omega').$$

Combining the formula for the iterated Gauss-Manin connection with the cup-product formula, we find that

$$\phi(\omega_A,\omega_B) : \frac{\partial}{\partial R_1}\otimes\cdots\otimes\frac{\partial}{\partial R_{2m-n}} \longmapsto c\operatorname{Res}_0\,\eta(R_1\cdots R_{2m-n}AB),$$

where $c \neq 0$ is fixed. Since the Grothendieck residue vanishes if and only if the argument of η lies in the Jacobian ideal, we see that

$$\phi(\omega_A,\omega_B)\text{ annihilates }\frac{\partial}{\partial R_1}\otimes\cdots\otimes\frac{\partial}{\partial R_{2m-n}}$$

if and only if $R_1 \cdots R_{2m-n} AB \in J_Q$. Since the products $R_1 \cdots R_{2m-n}$ generate V^s, where $s = (2m-n)q$, we see that

$$\phi(\omega_A, \omega_B) = 0 \Longleftrightarrow AB \in (J_Q : V^s).$$

More generally, let

$$\nu = \sum \nu_{ij}\, \omega_{A_i} \otimes \omega_{B_j}$$

be an arbitrary element of $S^2 F^m$. Then

$$\phi(\nu) = 0 \Longleftrightarrow \sum \nu_{ij} A_i B_j \in (J_Q : V^s).$$

This provides a description of the kernel of ϕ: Define an isomorphism from F^m to an appropriate space of adjoint polynomials,

$$\lambda : F^m \longrightarrow V^t$$

by

$$\lambda^{-1}(A) = \mathrm{res}(\Omega_A)$$

and consider the composition

$$\tilde{\mu} : S^2 F^m \xrightarrow{S^2\lambda} S^2 V^t \xrightarrow{\mu} V^{2t}$$

where the right-hand map is multiplication of polynomials. Then we have

$$\mathrm{kernel}(\phi) = \tilde{\mu}^{-1}((J_Q : V^s)^{2t}).$$

We now refine this assertion using Macaulay's theorem.

Proposition.

$$\mathrm{kernel}(\phi) = \tilde{\mu}^{-1}(J_Q^{2t}).$$

Proof. By Macaulay's theorem

$$(J_Q : V^s)^{2t} = (J_Q^{2t} + F^{\sigma - s + 1} V)^{2t}.$$

It therefore suffices to show that

$$\sigma - s + 1 > 2t. \tag{$*$}$$

now the degree of the adjoints is

$$t = (a+1)q - (n+2),$$

where $q = \deg Q$ and a is the least positive number for which $t \geq 0$. The highest level in the Hodge filtration is therefore $M = n - a$, and $2m - n = n - 2a$ derivatives are required to map

$$H^{n-a,a} \longrightarrow H^{a,n-a}.$$

Therefore $s = (n-a)q$ Putting all of this together, we see that $(*)$ is equivalent to $n + 3 > 0$, and so must hold automatically.

b. Construction of X from its Jacobian ideal

Let us now suppose that the vector space of polynomials J_Q^{2t} is known. Then by Macaulay's theorem we can reconstruct J_Q^{q-1}, hence J_Q itself:

Lemma. *If $L > n/2$, then*

$$J_Q^{q-1} = (J_Q^{2t} : V^{2t-q+1}).$$

Proof. By Macaulay's theorem it suffices to show that

$$\sigma - (2t - q + 1) + 1 > q - 1.$$

But this is equivalent to

$$a < \frac{n}{2} + \frac{n+3}{q},$$

which is certainly satisfied if $a < n/2$. Since $l = n - a$, the proof is complete.

By the lemma, we may assume J_Q^{q-1} is known as a vector space of polynomials. To construct Q from this datum, we appeal to the following result:

Lemma. *Let Q be a generic polynomial of degree at least three, and let x_i be a basis for V^1. Then there is a unique basis $F_0, \ldots, F_{n+1}$ for J_Q^{q-1}, up to a common nonzero multiple, such that*

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}.$$

Since $(\partial Q/\partial x_j)$ is a basis for J_Q^{q-1} satisfying the given condition, the lemma implies that

$$\sum x_j F_j = \lambda \sum x_j \frac{\partial Q}{\partial x_j} = \lambda' Q$$

whenever Q is generic:

Proposition. *If the degree exceeds two, then*

$$Q \longrightarrow J_Q^{q-1}$$

is generically injective as a map from a projective space to a Grassmannian.

Remark. The map $Q \mapsto J_Q^{q-1}$ is probably injective so long as $Q = 0$ does not possess singularities which are "large in relation to the degree".

We conclude this section with a proof of the Lemma. To begin, let (F_j) be an arbitrary basis for J_Q^{q-1}, and write

$$F_i = \sum A_{i\alpha} \frac{\partial Q}{\partial x_\alpha}.$$

The condition on the partials then yields

$$\sum A_{i\alpha} \frac{\partial^2 Q}{\partial x_j \partial x_\alpha} = \sum A_{j\beta} \frac{\partial^2 Q}{\partial x_i \partial x_\beta}$$

which we may write as

$$AH = H^t A, \qquad (**)$$

where

$$H = \left(\frac{\partial^2 Q}{\partial x_\alpha \partial x_\beta} \right)$$

is the Hessian matrix. The equality $(**)$ must hold for all $x \in \mathbb{C}^{n+2}$; we claim that for Q generic, this is possible only if A is a multiple of the identity which is what the lemma asserts.

Consider now the special case when

$$Q = \frac{1}{q(q-1)} \sum x_i^q$$

is the Fermat. Then the Hessian matrix is

$$H_{ij} = \delta_{ij} x_i^{q-2},$$

so that the equations $(**)$ become

$$A_{ij} x_i^{q-2} = A_{ji} x_i^{q-2}.$$

These imply that, for $q > 2$, $A_{ij} = 0$ if $i \neq j$, so that $A = (\mu_i)$ is a diagonal matrix. Thus, any basis of J_Q^{q-1} satisfying the hypothesis of the lemma has the form $F_j = \mu_j \partial Q / \partial x_j$, and

$$\sum x_j F_j = \lambda \left(\sum \mu_j x_j^q \right)$$

is projectively equivalent to Q.

Now let

$$Q(s) = Q + sQ'$$

represent a deformation of Q, and let

$$A(s) = A + sA' + \frac{s^2}{2} A'' + \cdots$$

be a family of matrices satisfying $(**)$ for all s near zero. Differentiating this relation and evaluating at $s = 0$, we obtain the additional condition

$$A'H + AH' = H'^t A + H^t A'$$

where H' is the hessian of Q'. Using what we have already found concerning A, this yields

$$A'_{ij}x_j^{q-2} + \mu_i H'_{ij} = H'_{ij}\mu_j + x_i^{q-2}A'_{ji}.$$

Suppose now that Q' is chosen so that H'_{ij} does not lie in the vector space spanned by x_i^{q-2} and x_j^{q-2}. Then we have

$$A'_{ij}x_j^{q-2} = x_i^{q-2}A'_{ji}$$

and

$$\mu_i H'_{ij} = H'_{ij}\mu_j.$$

The first equation implies that A' is diagonal, and the second implies that $\mu_i = \mu_j$. Consequently

$$A = \mu I + sA' + \frac{s^2}{2}A'' + \cdots$$

satisfies the desired conclusion to first order. Substitution back into the original equation (∗∗) yields

$$\left(A' + \frac{s}{2}A'' + \cdots\right)H(s) = H(s)\left({}^tA' + \frac{s}{2}\ {}^tA'' + \cdots\right).$$

Arguing as before, we find that A' is also a multiple of the identity, say $\mu' I$. Iteration of the argument then yields

$$A(s) = \mu(s)I$$

for the generic deformation of Q. We conclude that the only solution to (∗∗), for generic Q, is a multiple of the identity, as desired.

c. Construction of X from its local variation

It now remains to assemble the pieces: The kernel of the local variation of Hodge structure determines a piece of the Jacobian ideal by the formula

$$J_Q^{2t} = \tilde{\mu}(\text{kernel}(\phi)).$$

If this piece is nonzero, then Q is generically determined by the results of the preceding section. Unfortunately, to construct Q, we need not only to local variation Φ, but also the isomorphism

$$\lambda : F^m \longrightarrow V^t$$

provided by the residue map. Put another way, Q is determined by the pair (Φ, λ), where λ imposes a “polynomial structure” on F^l.

The proof of the Torelli theorem therefore reduces to the problem of defining λ in an *a priori* manner. In general we do not know how to do this. However, note that the set of all isomorphisms from F^m to V^t is homogeneous under the action of $GL(V^t)$. Moreover, it is not λ itself, but rather its $GL(V^1)$-orbit which is of significance: any isomorphism $GL(V^1)$-equivalent to λ produces, via the Euler formula, a variety $PGL(V^1)$-equivalent to $Q = 0$. Thus, if $t = 1$, any isomorphism will do, and hence a generic X is determined, up to $PGL(V^1)$-equivalence, by its local variation of Hodge structure.

Fortunately there is a class of hypersurfaces for which $t = 1$ and $J_Q^{2t} \neq 0$: these are exactly the cubics of dimension $3l$: It suffices to observe that

$$\Omega_A = \frac{A\Omega}{Q^{l+1}}$$

is homogeneous of degree zero if A is homogeneous of degree one. The proof of the main theorem is now complete.

Remarks. (1) The crucial feature of the cubic case is that the projective space inside which X sits is given naturally in terms of the Hodge structure: $V^1 \cong F^{2m}$, so that

$$X \subset \mathbb{P}((F^{2m})^*).$$

(2) To prove weak global Torelli in general, one must deduce the polynomial structure on F^m, up to $GL(V^1)$-equivalence, from the local variation of Hodge structure. One possible approach is suggested by the factorization

$$\tilde{\mu} = \mu \circ S^2\lambda$$

which precedes the proposition of the last setion. The factorization implies that

$$\text{kernel}(\phi) \supset (S^2\lambda)^{-1}(\text{kernel}(\mu)).$$

Consequently the kernel of ϕ contains a "fixed part" which depends on Q only through the polynomial structure of F^m. It is conceivable that this fixed part, which comes from a piece of the multiplication map and determines the Veronese imbedding of $\mathbb{P}\widehat{V}^t$) in $\mathbb{P}(\widehat{V}^{2t})$, could be defined Hodge-theoretically.

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