EXAMPLES

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In the following note, we work through theorem 2 of Langlands's essay *A little bit of number theory* for the primes 3 and 5. The theorem claims that two numbers A_p and B_p for p prime are equal. We show that $A_3 = B_3$ and $A_5 = B_5$. Later, we give the results of a computer program that calculates A_p and B_p for any p as a table for the first 500 odd primes.

We state the theorem:

Theorem.

(a) *If* $p \equiv 3 \pmod 4$ *then* $A_p = 0$. *If* $p \equiv 1 \pmod 4$ *write* $p = x^2 + y^2$ *with* (x, y) *congruent to* $(1, 0)$ *or to* $(-1, 2)$ *modulo* 4 *and set* $A_p = 2(x^3 - 3xy^2)$.

(b) *If* $p \equiv 3 \pmod 4$ *set*

$$B_p = \sum \{\alpha^2 + \beta^2 - \gamma^2 - \delta^2\} - \sum \{\alpha^2 + \beta^2 - \gamma^2 - \delta^2\}$$

The first sum is over 4*-tuples such that* $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = p$, $\alpha \equiv 1 \pmod 4$, *and* (β, γ, δ) *is congruent modulo* 4 *to one of* $(0, 1, 1)$, $(2, 3, 3)$, $(0, 3, 3)$, $(2, 1, 1)$. *The second sum is similar but now* (β, γ, δ) *is congruent to one of* $(0, 3, 1)$, $(2, 1, 3)$, $(0, 1, 3)$, $(2, 3, 1)$.

But if $p \equiv 1 \pmod 4$ *set*

$$B_p = \sum \{\alpha^2 + \beta^2 - \gamma^2 - \delta^2\} - \sum \{\alpha^2 + \beta^2 - \gamma^2 - \delta^2\}$$

The two sums are defined similarly. Again $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = p$ *and* $\alpha \equiv 1 \pmod 4$, *but in the first sum* (β, γ, δ) *is congruent modulo* 4 *to one of* $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 2)$, $(2, 2, 2)$ *and in the second to one of* $(2, 2, 0)$, $(0, 2, 0)$, $(2, 0, 2)$, $(0, 0, 2)$.

1. $A_3 = B_3$

Let $p = 3$ and

$$A = \sum \left\{\alpha^2 + \beta^2 - \gamma^2 - \delta^2\right\},$$

where this sum is the first one in the definition of B_3. We define the number B similarly. Then, $B_3 = A - B$.

We calculate B_3 first. To determine the ranges of summation of A and B, we list the possible decompositions of 3 as a sum of four squares subject to the constraints of the theorem. It is clear that $\alpha = 1$, and so we obtain the following:

$$\begin{aligned} 3 &= 1^2 + 0^2 + 1^2 + 1^2 && (0, 1, 1) \\ &= 1^2 + 0^2 + (-1)^2 + (-1)^2 && (0, 3, 3) \\ &= 1^2 + 0^2 + (-1)^2 + 1^2 && (0, 3, 1) \\ &= 1^2 + 0^2 + 1^2 + (-1)^2 && (0, 1, 3) \end{aligned}$$

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For each triple (b, g, d) on the right, $(\beta, \gamma, \delta) \equiv (b, g, d) \pmod 4$. We calculate B_3:

$$\begin{aligned} A &= 1^2 + 0^2 - 1^2 - 1^2 + \\ &\quad\ 1^2 + 0^2 - (-1)^2 - (-1)^2 \\ &= -2. \\ B &= 1^2 + 0^2 - (-1)^2 - 1^2 + \\ &\quad\ 1^2 + 0^2 - 1^2 - (-1)^2 \\ &= -2. \end{aligned}$$

It follows that $B_3 = A - B = 0$. By part (a), since $3 \equiv 3 \pmod 4$, $A_3 = 0$. $A_3 = B_3 = 0$.

2. $A_5 = B_5$

Since $5 \equiv 1 \pmod 4$, we write $5 = x^2 + y^2 = (-1)^2 + 2^2$. Then,

$$A_5 = 2(x^3 - 3xy^2) = 2((-1)^3 - 3(-1)(2^2)) = 2(-1 + 12) = 22.$$

Using A and B defined in the previous section, we calculate B_5.

$$\begin{aligned} A &= 1^2 + 2^2 - 0^2 - 0^2 + && (2,0,0) \\ &\quad\ 1^2 + (-2)^2 - 0^2 - 0^2 \\ &= 10 \\ B &= 1^2 + 0^2 - 2^2 - 0^2 + && (0,2,0) \\ &\quad\ 1^2 + 0^2 - (-2)^2 - 0^2 + \\ &\quad\ 1^2 + 0^2 - 0^2 - 2^2 + && (0,0,2) \\ &\quad\ 1^2 + 0^2 - 0^2 - (-2)^2 \\ &= -3 \cdot 4 = -12. \end{aligned}$$

Finally, $B_5 = A - B = 10 - (-12) = 22$. It follows that $A_5 = B_5$.

3. Program

The Haskell program `lbnt.hs` calculates A_p and B_p for the first 500 odd primes. It can be easily modified to calculate A_p and B_p any number of primes. Please see the comments in the source code.

The result follows:

nth odd prime	p	$p \pmod 4$	A_p	B_p
1	3	3	0	0
2	5	1	22	22
3	7	3	0	0
4	11	3	0	0
5	13	1	−18	−18
6	17	1	−94	−94
7	19	3	0	0
8	23	3	0	0
9	29	1	−130	−130
10	31	3	0	0
11	37	1	214	214

12	41	1	−230	−230
13	43	3	0	0
14	47	3	0	0
15	53	1	518	518
16	59	3	0	0
17	61	1	830	830
18	67	3	0	0
19	71	3	0	0
20	73	1	1098	1098
21	79	3	0	0
22	83	3	0	0
23	89	1	−1670	−1670
24	97	1	594	594
25	101	1	598	598
26	103	3	0	0
27	107	3	0	0
28	109	1	−1746	−1746
29	113	1	2002	2002
30	127	3	0	0
31	131	3	0	0
32	137	1	−1606	−1606
33	139	3	0	0
34	149	1	−3514	−3514
35	151	3	0	0
36	157	1	286	286
37	163	3	0	0
38	167	3	0	0
39	173	1	−4082	−4082
40	179	3	0	0
41	181	1	3942	3942
42	191	3	0	0
43	193	1	5362	5362
44	197	1	1174	1174
45	199	3	0	0
46	211	3	0	0
47	223	3	0	0
48	227	3	0	0
49	229	1	6390	6390
50	233	1	−598	−598
51	239	3	0	0
52	241	1	−5310	−5310
53	251	3	0	0
54	257	1	−1534	−1534
55	263	3	0	0
56	269	1	3406	3406
57	271	3	0	0
58	277	1	9126	9126
59	281	1	−7430	−7430

60	283	3	0	0
61	293	1	−9418	−9418
62	307	3	0	0
63	311	3	0	0
64	313	1	−6838	−6838
65	317	1	−10274	−10274
66	331	3	0	0
67	337	1	−12366	−12366
68	347	3	0	0
69	349	1	9470	9470
70	353	1	3298	3298
71	359	3	0	0
72	367	3	0	0
73	373	1	−12922	−12922
74	379	3	0	0
75	383	3	0	0
76	389	1	374	374
77	397	1	9614	9614
78	401	1	−2398	−2398
79	409	1	7146	7146
80	419	3	0	0
81	421	1	−10890	−10890
82	431	3	0	0
83	433	1	−4862	−4862
84	439	3	0	0
85	443	3	0	0
86	449	1	16114	16114
87	457	1	16506	16506
88	461	1	2318	2318
89	463	3	0	0
90	467	3	0	0
91	479	3	0	0
92	487	3	0	0
93	491	3	0	0
94	499	3	0	0
95	503	3	0	0
96	509	1	14270	14270
97	521	1	23738	23738
98	523	3	0	0
99	541	1	−5922	−5922
100	547	3	0	0
101	557	1	−8626	−8626
102	563	3	0	0
103	569	1	−26806	−26806
104	571	3	0	0
105	577	1	−3454	−3454
106	587	3	0	0
107	593	1	−15502	−15502

108	599	3	0	0
109	601	1	−17030	−17030
110	607	3	0	0
111	613	1	23222	23222
112	617	1	15466	15466
113	619	3	0	0
114	631	3	0	0
115	641	1	28850	28850
116	643	3	0	0
117	647	3	0	0
118	653	1	33358	33358
119	659	3	0	0
120	661	1	−25850	−25850
121	673	1	−4462	−4462
122	677	1	4054	4054
123	683	3	0	0
124	691	3	0	0
125	701	1	20030	20030
126	709	1	−36810	−36810
127	719	3	0	0
128	727	3	0	0
129	733	1	38718	38718
130	739	3	0	0
131	743	3	0	0
132	751	3	0	0
133	757	1	35046	35046
134	761	1	31882	31882
135	769	1	9650	9650
136	773	1	39542	39542
137	787	3	0	0
138	797	1	−41954	−41954
139	809	1	−23270	−23270
140	811	3	0	0
141	821	1	−1850	−1850
142	823	3	0	0
143	827	3	0	0
144	829	1	23166	23166
145	839	3	0	0
146	853	1	−20378	−20378
147	857	1	45994	45994
148	859	3	0	0
149	863	3	0	0
150	877	1	−42514	−42514
151	881	1	−7150	−7150
152	883	3	0	0
153	887	3	0	0
154	907	3	0	0
155	911	3	0	0

156	919	3	0	0
157	929	1	30866	30866
158	937	1	51946	51946
159	941	1	−31378	−31378
160	947	3	0	0
161	953	1	−56758	−56758
162	967	3	0	0
163	971	3	0	0
164	977	1	−56606	−56606
165	983	3	0	0
166	991	3	0	0
167	997	1	52886	52886
168	1009	1	63810	63810
169	1013	1	−42458	−42458
170	1019	3	0	0
171	1021	1	−56738	−56738
172	1031	3	0	0
173	1033	1	18378	18378
174	1039	3	0	0
175	1049	1	−30470	−30470
176	1051	3	0	0
177	1061	1	40982	40982
178	1063	3	0	0
179	1069	1	65806	65806
180	1087	3	0	0
181	1091	3	0	0
182	1093	1	−71082	−71082
183	1097	1	4234	4234
184	1103	3	0	0
185	1109	1	41350	41350
186	1117	1	66654	66654
187	1123	3	0	0
188	1129	1	25434	25434
189	1151	3	0	0
190	1153	1	59202	59202
191	1163	3	0	0
192	1171	3	0	0
193	1181	1	34430	34430
194	1187	3	0	0
195	1193	1	−75478	−75478
196	1201	1	−55150	−55150
197	1213	1	−39042	−39042
198	1217	1	−11966	−11966
199	1223	3	0	0
200	1229	1	84910	84910
201	1231	3	0	0
202	1237	1	60966	60966
203	1249	1	85410	85410

204	1259	3	0	0
205	1277	1	−73634	−73634
206	1279	3	0	0
207	1283	3	0	0
208	1289	1	−72310	−72310
209	1291	3	0	0
210	1297	1	−7774	−7774
211	1301	1	70150	70150
212	1303	3	0	0
213	1307	3	0	0
214	1319	3	0	0
215	1321	1	−38630	−38630
216	1327	3	0	0
217	1361	1	14818	14818
218	1367	3	0	0
219	1373	1	−100418	−100418
220	1381	1	−97290	−97290
221	1399	3	0	0
222	1409	1	−86350	−86350
223	1423	3	0	0
224	1427	3	0	0
225	1429	1	−99866	−99866
226	1433	1	87098	87098
227	1439	3	0	0
228	1447	3	0	0
229	1451	3	0	0
230	1453	1	−25938	−25938
231	1459	3	0	0
232	1471	3	0	0
233	1481	1	−31990	−31990
234	1483	3	0	0
235	1487	3	0	0
236	1489	1	−7326	−7326
237	1493	1	−59962	−59962
238	1499	3	0	0
239	1511	3	0	0
240	1523	3	0	0
241	1531	3	0	0
242	1543	3	0	0
243	1549	1	17710	17710
244	1553	1	116978	116978
245	1559	3	0	0
246	1567	3	0	0
247	1571	3	0	0
248	1579	3	0	0
249	1583	3	0	0
250	1597	1	127134	127134
251	1601	1	−9598	−9598

252	1607	3	0	0
253	1609	1	28746	28746
254	1613	1	108238	108238
255	1619	3	0	0
256	1621	1	95238	95238
257	1627	3	0	0
258	1637	1	−66154	−66154
259	1657	1	134026	134026
260	1663	3	0	0
261	1667	3	0	0
262	1669	1	−123210	−123210
263	1693	1	−29378	−29378
264	1697	1	133906	133906
265	1699	3	0	0
266	1709	1	−15890	−15890
267	1721	1	102938	102938
268	1723	3	0	0
269	1733	1	137462	137462
270	1741	1	107822	107822
271	1747	3	0	0
272	1753	1	126522	126522
273	1759	3	0	0
274	1777	1	−58734	−58734
275	1783	3	0	0
276	1787	3	0	0
277	1789	1	52670	52670
278	1801	1	35210	35210
279	1811	3	0	0
280	1823	3	0	0
281	1831	3	0	0
282	1847	3	0	0
283	1861	1	−107818	−107818
284	1867	3	0	0
285	1871	3	0	0
286	1873	1	−83358	−83358
287	1877	1	−89626	−89626
288	1879	3	0	0
289	1889	1	−153374	−153374
290	1901	1	−56210	−56210
291	1907	3	0	0
292	1913	1	−142502	−142502
293	1931	3	0	0
294	1933	1	133198	133198
295	1949	1	133214	133214
296	1951	3	0	0
297	1973	1	−174938	−174938
298	1979	3	0	0
299	1987	3	0	0

300	1993	1	−121862	−121862
301	1997	1	152366	152366
302	1999	3	0	0
303	2003	3	0	0
304	2011	3	0	0
305	2017	1	−103086	−103086
306	2027	3	0	0
307	2029	1	−181170	−181170
308	2039	3	0	0
309	2053	1	170102	170102
310	2063	3	0	0
311	2069	1	185350	185350
312	2081	1	39442	39442
313	2083	3	0	0
314	2087	3	0	0
315	2089	1	164970	164970
316	2099	3	0	0
317	2111	3	0	0
318	2113	1	−130878	−130878
319	2129	1	196466	196466
320	2131	3	0	0
321	2137	1	−176726	−176726
322	2141	1	63230	63230
323	2143	3	0	0
324	2153	1	−72742	−72742
325	2161	1	167490	167490
326	2179	3	0	0
327	2203	3	0	0
328	2207	3	0	0
329	2213	1	206518	206518
330	2221	1	−129330	−129330
331	2237	1	−136994	−136994
332	2239	3	0	0
333	2243	3	0	0
334	2251	3	0	0
335	2267	3	0	0
336	2269	1	98494	98494
337	2273	1	−189598	−189598
338	2281	1	113130	113130
339	2287	3	0	0
340	2293	1	−219098	−219098
341	2297	1	206986	206986
342	2309	1	179446	179446
343	2311	3	0	0
344	2333	1	34142	34142
345	2339	3	0	0
346	2341	1	−183690	−183690
347	2347	3	0	0

348	2351	3	0	0
349	2357	1	28454	28454
350	2371	3	0	0
351	2377	1	−225414	−225414
352	2381	1	−157010	−157010
353	2383	3	0	0
354	2389	1	233350	233350
355	2393	1	−126022	−126022
356	2399	3	0	0
357	2411	3	0	0
358	2417	1	230594	230594
359	2423	3	0	0
360	2437	1	−224714	−224714
361	2441	1	−229622	−229622
362	2447	3	0	0
363	2459	3	0	0
364	2467	3	0	0
365	2473	1	−175318	−175318
366	2477	1	−227506	−227506
367	2503	3	0	0
368	2521	1	186410	186410
369	2531	3	0	0
370	2539	3	0	0
371	2543	3	0	0
372	2549	1	−104314	−104314
373	2551	3	0	0
374	2557	1	248094	248094
375	2579	3	0	0
376	2591	3	0	0
377	2593	1	−225182	−225182
378	2609	1	−94846	−94846
379	2617	1	−260406	−260406
380	2621	1	−162338	−162338
381	2633	1	43258	43258
382	2647	3	0	0
383	2657	1	160034	160034
384	2659	3	0	0
385	2663	3	0	0
386	2671	3	0	0
387	2677	1	−151866	−151866
388	2683	3	0	0
389	2687	3	0	0
390	2689	1	−244926	−244926
391	2693	1	71158	71158
392	2699	3	0	0
393	2707	3	0	0
394	2711	3	0	0
395	2713	1	48618	48618

396	2719	3	0	0
397	2729	1	−80870	−80870
398	2731	3	0	0
399	2741	1	286150	286150
400	2749	1	−73186	−73186
401	2753	1	112882	112882
402	2767	3	0	0
403	2777	1	−288086	−288086
404	2789	1	245174	245174
405	2791	3	0	0
406	2797	1	205326	205326
407	2801	1	117698	117698
408	2803	3	0	0
409	2819	3	0	0
410	2833	1	293618	293618
411	2837	1	146534	146534
412	2843	3	0	0
413	2851	3	0	0
414	2857	1	−186966	−186966
415	2861	1	−271282	−271282
416	2879	3	0	0
417	2887	3	0	0
418	2897	1	300514	300514
419	2903	3	0	0
420	2909	1	−265954	−265954
421	2917	1	17494	17494
422	2927	3	0	0
423	2939	3	0	0
424	2953	1	251962	251962
425	2957	1	319406	319406
426	2963	3	0	0
427	2969	1	−253894	−253894
428	2971	3	0	0
429	2999	3	0	0
430	3001	1	−142902	−142902
431	3011	3	0	0
432	3019	3	0	0
433	3023	3	0	0
434	3037	1	−189794	−189794
435	3041	1	−327470	−327470
436	3049	1	−94230	−94230
437	3061	1	320870	320870
438	3067	3	0	0
439	3079	3	0	0
440	3083	3	0	0
441	3089	1	−311630	−311630
442	3109	1	−46154	−46154
443	3119	3	0	0

444	3121	1	255762	255762
445	3137	1	−18814	−18814
446	3163	3	0	0
447	3167	3	0	0
448	3169	1	−285230	−285230
449	3181	1	129870	129870
450	3187	3	0	0
451	3191	3	0	0
452	3203	3	0	0
453	3209	1	170554	170554
454	3217	1	−167886	−167886
455	3221	1	268070	268070
456	3229	1	−365634	−365634
457	3251	3	0	0
458	3253	1	−369018	−369018
459	3257	1	204314	204314
460	3259	3	0	0
461	3271	3	0	0
462	3299	3	0	0
463	3301	1	29302	29302
464	3307	3	0	0
465	3313	1	348498	348498
466	3319	3	0	0
467	3323	3	0	0
468	3329	1	−374350	−374350
469	3331	3	0	0
470	3343	3	0	0
471	3347	3	0	0
472	3359	3	0	0
473	3361	1	275490	275490
474	3371	3	0	0
475	3373	1	−60498	−60498
476	3389	1	100670	100670
477	3391	3	0	0
478	3407	3	0	0
479	3413	1	−140602	−140602
480	3433	1	398682	398682
481	3449	1	253786	253786
482	3457	1	334386	334386
483	3461	1	−405418	−405418
484	3463	3	0	0
485	3467	3	0	0
486	3469	1	207630	207630
487	3491	3	0	0
488	3499	3	0	0
489	3511	3	0	0
490	3517	1	398014	398014
491	3527	3	0	0

492	3529	1	398090	398090
493	3533	1	257998	257998
494	3539	3	0	0
495	3541	1	406150	406150
496	3547	3	0	0
497	3557	1	104566	104566
498	3559	3	0	0
499	3571	3	0	0
500	3581	1	375358	375358