Dirichlet character mod q, $\chi(n)=0$ for $(n,q)>1$

$\chi(n)=\chi(m)$ if $n\equiv m \pmod q$

totally multiplicative

$$\chi(m)\chi(n)=\chi(mn).$$

$\chi(1)=1$. If $q=\prod_i p_i^{\alpha_i}$, then

a character χ mod q has a unique factorization

(1) $$\chi(n)=\prod_i \chi_i(n),$$

where χ_i is a character mod $p_i^{\alpha_i}$.

If $\chi_i(n)$ for $(n,p_i)=1$ does not coinside with a character mod $p_i^{\alpha_i-1}$, we say that χ_i is a primitive character mod $p_i^{\alpha_i}$. Finally if in (1) every χ_i is primitive mod $p_i^{\alpha_i}$, we say χ is a primitive character mod q.

The number of distinct characters mod q is $\varphi(q)=q\prod_{p/q}(1-\frac{1}{p})$ and the number of primitive characters is

$$\varphi^*(q)=q\prod_{\substack{p/q\\ p^2\nmid q}}\left(1-\frac{2}{p}\right)\prod_{p^2/q}\left(1-\frac{1}{p}\right)^2$$

The character $\chi(n)=1$ if $(n,q)=1$ is called the principal character $\pmod q$ and denoted by $\chi_0(n)$. We have $\sum_{n \bmod q}\chi(n)=\begin{cases}0 & \text{for } \chi\neq\chi_0\\ \varphi(q) & \text{for } \chi=\chi_0.\end{cases}$

We also have

$$\sum_{\chi} \chi(n) = \begin{cases} 0 & \text{for } n \not\equiv 1 \pmod q \\ \varphi(q) & \text{for } n \equiv 1 \pmod q. \end{cases}$$

Gauss or Jacobi-sums

Define

$$(2) \qquad \tau_\chi(n) = \sum_{\ell \bmod q} \chi(\ell) e^{2\pi i \frac{n\ell}{q}}.$$

If χ is a primitive character mod q and $(n,q) > 1$ then $\tau_\chi(n) = 0$.
(Proof: if $(n,q) > 1$ there is a $p \mid (n,q)$, combine terms where ℓ belongs to same residue class mod $\frac{q}{p}$, and show sum is zero)

If $(n,q) = 1$, then

$$\tau_\chi(n) = \bar{\chi}(n) \sum_{\ell(q)} \chi(n\ell) e^{2\pi i \frac{n\ell}{q}} = \bar{\chi}(n) \tau_\chi(1)$$

or writing τ_χ for $\tau_\chi(1)$, $\tau_\chi(n) = \bar{\chi}(n)\tau_\chi$

Thus

$$\sum_{n(q)} |\tau_\chi(n)|^2 = \varphi(q) |\tau_\chi|^2$$

But also

$$\sum_{n(q)} |\tau_\chi(n)|^2 = \sum_{n(q)} \sum_{\ell,k} \chi(\ell)\bar{\chi}(k) e^{2\pi i \frac{n(\ell-k)}{q}}$$

$$= q \sum_{\ell} |\chi(\ell)|^2 = q\,\varphi(q)$$

So $|\tau_\chi|^2 = q$. As $\bar{\tau}_\chi = \chi(-1)\tau_{\bar{\chi}}$, we get $\tau_\chi \tau_{\bar{\chi}} = \chi(-1) q$.

Poisson Summation formula:

If $\hat{f}(v) = \int_{-\infty}^{\infty} f(u)\, e^{2\pi i u v}\, du$,

then (under conditions always satisfied in the cases when we use the formula later) we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

There is a generalization, if χ is a primitive character mod q, then

$$(3)\quad \varepsilon_\chi \sum_n \chi(n) f\left(\frac{n}{\sqrt{q}}\right) = \bar{\varepsilon}_\chi \sum_n \bar{\chi}(n) \hat{f}\left(\frac{n}{\sqrt{q}}\right),$$

where $|\varepsilon_\chi| = 1$.

We have

$$\sum_n \chi(n) f\left(\frac{n}{q}\right) = \sum_{l(q)} \chi(l) \sum_n f\left(\frac{qn+l}{\sqrt{q}}\right).$$

From classical Poisson formula

$$\sum_n f\left(\frac{qn+l}{\sqrt{q}}\right) = \frac{1}{\sqrt{q}} \sum_m \hat{f}\left(\frac{m}{\sqrt{q}}\right) e^{-2\pi i \frac{lm}{q}},$$

so

$$\sum_n \chi(n) f\left(\frac{n}{\sqrt{q}}\right) = \frac{1}{\sqrt{q}} \sum_m \hat{f}\left(\frac{m}{\sqrt{q}}\right) \tau_\chi(-m)$$

$$= \frac{\chi(-1)\tau_\chi}{\sqrt{q}} \sum_m \bar{\chi}(m) \hat{f}\left(\frac{m}{\sqrt{q}}\right),$$

which proves (3) with $\varepsilon_\chi^{-2} = \frac{\chi(-1)\tau_\chi}{\sqrt{q}}$.

Poisson Summation formula:

If $\hat{f}(v) = \int_{-\infty}^{\infty} f(u)\, e^{2\pi i u v}\, du$

Then (under suitable conditions)

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

There is a generalization if χ is a primitive character mod q, then

$$(3) \quad \varepsilon_\chi \sum_n \chi(n) f\left(\frac{n}{\sqrt{q}}\right) = \varepsilon_{\bar\chi} \sum_n \bar\chi(n) \hat{f}\left(\frac{n}{\sqrt{q}}\right),$$

where $\bar\varepsilon_\chi = \varepsilon_{\bar\chi}$ and $|\varepsilon_\chi| = 1$.

We have

$$\sum_n \chi(n) f\left(\frac{n}{\sqrt{q}}\right) = \sum_{\ell\,(q)} \chi(\ell) \sum_n f\left(\frac{qn+\ell}{\sqrt{q}}\right).$$

From classical Poisson formula

$$\sum_n f\left(\frac{qn+\ell}{\sqrt{q}}\right) = \frac{1}{\sqrt{q}} \sum_n \hat{f}\left(\frac{n}{\sqrt{q}}\right) e^{-2\pi i \frac{\ell n}{q}},$$

So

$$\sum_n \chi(n) f\left(\frac{n}{\sqrt{q}}\right) = \frac{1}{\sqrt{q}} \sum_n \hat{f}\left(\frac{n}{\sqrt{q}}\right) \tau_\chi(-n)$$

$$= \frac{\chi(-1)\tau_\chi}{\sqrt{q}} \sum_n \bar\chi(n) \hat{f}\left(\frac{n}{\sqrt{q}}\right),$$

which proves (3) with $\varepsilon_{\bar\chi}^2 = \frac{\chi(-1)\tau_\chi}{\sqrt{q}}$

For $s = \sigma + it$; $\sigma > 1$ define

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_p (1-\chi(p)p^{-s})^{-1},$$

we assume χ is a primitive character mod q; and include the case $q=1$, with $\chi(n)=1$ for all integers n. In that case we may write $\zeta(s)$ instead of $L(s,\chi)$.

If we want to extend the domain of definition beyond $\sigma \geq 1$, we can write $L(s,\chi)$ as a Stieltjes integral

$$L(s,\chi) = \int_{\frac{1}{2}}^{\infty} \frac{d\, s_\chi(x)}{x^s} \; ;$$

where we have written

$$\sum_{n \leq x} \chi(n) .$$

Then by partial integration

$$L(s,\chi) = \left[\frac{s_\chi(x)}{x^s} \right]_{\frac{1}{2}}^{\infty} + s \int_1^{\infty} \frac{s_\chi(x)}{x^{s+1}} dx$$

$$= s \int_1^{\infty} \frac{s_\chi(x)}{x^{s+1}} dx .$$

For $q > 1$ $|s_\chi(x)|$ is bounded $< q$. So $L(s,\chi)$ is regular for $\sigma > 0$ and

$|L(s,\chi)| < A_\delta\, q\, (1+|t|)$ for $\sigma > \delta > 0$.

If $q=1$ we have $s_\chi(x)=[x]$, so

$$\zeta(s)=\sum_{n=1}^{\infty} n^{-s} = s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx =$$

$$= s\int_1^\infty \frac{dx}{x^s} - s\int_1^\infty \frac{x-[x]}{x^s}\,dx =$$

$$= \frac{s}{s-1} - s\int_1^\infty \frac{x-[x]}{x^s}\,dx$$

Thus $\zeta(s)-\frac{1}{s-1}$ is regular for $\sigma>\delta>0$ and bounded by $A_\delta(1+|t|)$.

To effect analytic continuation of these functions in the whole complex plane, we go back to the generalized Poisson formula (3). We have

If
$$f(u)=x^{\frac14}e^{-\pi u^2 x} \text{ then } \hat f(u)=x^{-\frac14}e^{-\pi u^2\frac1x},$$

and if
$$g(u)=x^{\frac34}u\,e^{-\pi u^2 x} \text{ then } \hat g(u)=i\,x^{-\frac34}u\,e^{-\pi u^2\frac1x},$$

If $\chi(-1)=1$, we put the above $f(u)$, $\hat f(u)$ in (3) and get for $q>1$ since f and $\hat f$ are even

$$\text{(4)}\quad \varepsilon_\chi \sum_{n=1}^{\infty} \chi(n)\, x^{\frac14} e^{-\pi\frac{n^2}{q}x} = \bar\varepsilon_\chi \sum_{n=1}^{\infty} \bar\chi(n)\, x^{-\frac14} e^{-\pi\frac{n^2}{q}\frac1x}.$$

which we may rewrite as

$$(4') \quad \varepsilon_\chi \, x^{\frac{1}{4}} \, \theta_\chi(x) = \bar{\varepsilon}_\chi \, x^{-\frac{1}{4}} \, \theta_{\bar{\chi}}\left(\frac{1}{x}\right),$$

where $\theta_\chi(x) = \sum_1^\infty \chi(n) e^{-\pi \frac{n^2}{q} x}$.

For an odd character when $\chi(-1) = -1$, we put the functions g and $\hat{g}$ in (3) and get when we multiply by $\sqrt{q}$,

$$(5) \quad \varepsilon_\chi \sum_1^\infty \chi(n) x^{\frac{3}{4}} n e^{-\pi \frac{n^2}{q} x} =$$

$$= i \bar{\varepsilon}_\chi \sum_1^\infty \bar{\chi}(n) \, n \, x^{-\frac{3}{4}} e^{-\pi \frac{n^2}{q} \frac{1}{x}},$$

or

$$(5') \quad \varepsilon_\chi \, x^{\frac{3}{4}} \, \theta_\chi^*(x) = i \bar{\varepsilon}_\chi \, x^{-\frac{3}{4}} \, \theta_{\bar{\chi}}^*\left(\frac{1}{x}\right),$$

where we have put

$$\theta_\chi^*(x) = \sum_{n=1}^\infty \chi(n) n e^{-\pi \frac{n^2}{q} x}.$$

For $q = 1$, we get putting f and $\hat{f}$ in the classical Poisson formula, that

$$(6) \quad x^{\frac{1}{4}}\left(1 + 2\sum_{n=1}^\infty e^{-\pi n^2 x}\right) = x^{-\frac{1}{4}}\left(1 + 2\sum_{n=1}^\infty e^{-\pi n^2 \frac{1}{x}}\right),$$

here the presence of the constant term in the brackets will cause some difficulty.

We have for $a > 0$; $\sigma > 0$

$$\int_0^\infty x^s e^{-ax} \frac{dx}{x} = \frac{\Gamma(s)}{a^s} .$$

Consider first case $q > 1$; χ even, we have for $\sigma > 1$

$$\int_0^\infty x^{\frac{s}{2}} \theta_\chi(x) \frac{dx}{x} = \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\tfrac{s}{2}\right) \sum_{n=1}^\infty \chi(n) n^{-s} =$$

$$= \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\tfrac{s}{2}\right) L(s, \chi) .$$

Here the integral on the lefthandside exists for all complex s and shows that $\pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi)$ is an integral function, and consequently $L(s,\chi)$ is an integral function.

Furthermore we have by (4').

$$\varepsilon_\chi \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\tfrac{s}{2}\right) L(s,\chi) = \varepsilon_\chi \int_0^\infty x^{\frac{s}{2}-\frac{1}{4}} \, x^{\frac{1}{4}} \theta_\chi(x) \frac{dx}{x}$$

$$= \bar{\varepsilon}_\chi \int_0^\infty x^{\frac{s}{2}-\frac{1}{4}} \, x^{-\frac{1}{4}} \theta_{\bar{\chi}}\left(\tfrac{1}{x}\right) \frac{dx}{x} = \bar{\varepsilon}_\chi \int_0^\infty x^{\frac{1-s}{2}} \theta_{\bar{\chi}}(x) \frac{dx}{x}$$

$$= \bar{\varepsilon}_\chi \pi^{\frac{s-1}{2}} q^{\frac{1-s}{2}} \Gamma\left(\tfrac{1-s}{2}\right) L(1-s, \bar{\chi}) \text{ , or,}$$

(7) $$\varepsilon_\chi \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\tfrac{s}{2}\right) L(s,\chi) = \bar{\varepsilon}_\chi \pi^{\frac{s-1}{2}} q^{\frac{1-s}{2}} \Gamma\left(\tfrac{1-s}{2}\right) L(1-s, \bar{\chi})$$

We now consider the case when χ is odd, and consider

$$\sqrt{\frac{\pi}{q}} \int_0^\infty x^{\frac{s+1}{2}} \theta_\chi^*(x) \frac{dx}{x} = \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) \sum_{n=1}^\infty \chi(n) n^{-s}$$

$$= \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi).$$

Using now (5') in the same way as we before used (4'), we get further for $\chi(-1) = -1$, that

(8) $$\varepsilon_\chi \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) =$$
$$= i\,\overline{\varepsilon}_\chi\, \pi^{\frac{s-1}{2}} q^{\frac{1-s}{2}} \Gamma\left(\frac{2-s}{2}\right) L(1-s,\overline{\chi}),$$

writing $\varepsilon'_\chi = e^{-i\frac{\pi}{4}} \varepsilon_\chi$, this takes the form

(8') $$\varepsilon'_\chi \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) =$$
$$= \overline{\varepsilon}'_\chi\, \pi^{\frac{s-1}{2}} q^{\frac{1-s}{2}} \Gamma\left(\frac{2-s}{2}\right) L(1-s,\overline{\chi}).$$

Consequences:

Writing $\xi(s,\chi)$ for the lefthand-side of (7) and (8'), the functional equation can be written in the form

(9) $$\xi(s,\chi) = \overline{\xi(1-\overline{s},\chi)},$$

We can conclude that $\xi(s,x)$ is an integral function of s, which is real on the line $\sigma = \frac{1}{2}$. All zeros of $\xi(s,x)$ lie in the strip $0 \le \sigma \le 1$.

We still need to deal with the case $q=1$, that is $\xi(s)$. Formula

(6) $$x^{\frac{1}{4}}\left(1+2\sum_{n=1}^{\infty} e^{-\pi n^2 x}\right) =$$

$$= x^{-\frac{1}{4}}\left(1+2\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{x}}\right)$$

can not be dealt with in the same way because of the presence of the constant terms in the brackets. The two operators $x^{\frac{5}{4}}\frac{d}{dx}x^{-\frac{1}{4}}$ and $x^{\frac{3}{4}}\frac{d}{dx}x^{-\frac{1}{4}}$ commute (actually they are both of the form $x^{1-\alpha}\frac{d}{dx}x^{\alpha} = x\frac{d}{dx}+\alpha$), their product $D = \left(x\frac{d}{dx}\right)^2 - \frac{1}{16}$ will annihilate the constant terms on both sides of (6). D is also easily seen to be selfadjoint with respect to the measure $\frac{dx}{x}$.

Writing $\theta(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x}$,

we now consider, first for $\sigma > 1$,

$$\int_0^\infty x^{\frac{s}{2}-\frac{1}{4}} D(x^{\frac{1}{4}}\theta(x)) \frac{dx}{x} =$$

$$= 2\sum_{n=1}^{\infty} \int_0^\infty x^{s-\frac{1}{4}} D(x^{\frac{1}{4}} e^{-\pi n^2 x}) \frac{dx}{x}.$$

Here for $a > 0$

$$\int_0^\infty x^{\frac{s}{2}-\frac{1}{4}} D(x^{\frac{1}{4}} e^{-ax}) \frac{dx}{x} =$$

$$= \int_0^\infty x^{\frac{1}{4}} e^{-ax} D(x^{\frac{s}{2}-\frac{1}{4}}) \frac{dx}{x} =$$

$$= \frac{s(s-1)}{4} \int_0^\infty x^{\frac{s}{2}} e^{-ax} \frac{dx}{x} =$$

$$= \frac{s(s-1)}{4} \Gamma\left(\frac{s}{2}\right) a^{-\frac{s}{2}}.$$

Thus

$$(10) \quad \int_0^\infty x^{\frac{s}{2}-\frac{1}{4}} D(x^{\frac{1}{4}}\Theta(x)) \frac{dx}{x}$$

$$= \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} n^{-s} =$$

$$= \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$