$$\xi(s,\chi) = \overline{\xi(1-\bar{s},\chi)}$$

for χ primitive and even, where $q>0$ and

$$\xi(s,\chi) = \varepsilon_\chi \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s,\chi),$$

and for χ odd

$$\xi(s,\chi) = \varepsilon'_\chi \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi),$$

and finally for the case $q=1$

$$\xi(s) = \xi(1-s),$$

where

$$\xi(s) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The ξ are all integral functions with all of their zeros in the strip $0 \le \sigma \le 1$, and symmetrically to the line $\sigma = \frac{1}{2}$.

Lemma. If $f(z)$ is analytic in $|z| \le R$; $f(0) \ne 0$ and $|f(z)| < M$ for $|z| \le R$, then the number of zeros of $f(z)$ in the circle $|z| \le r$ where $r < R$

is bounded by

$$\frac{\log \frac{M}{|f(0)|}}{\log \frac{R}{r}}.$$

Denote the zeros of $f(z)$ in the circle $|z| \le r$ by α_i and write $r_i = |\alpha_i|$,

write
$$g(z) = \prod_i \frac{z - \alpha_i}{R - \frac{\bar{\alpha}_i z}{R}};$$

We have $|g(0)| = \prod_i \frac{r_i}{R} \le \left(\frac{r}{R}\right)^n$
if n is the number of zeros.
Since $\frac{f(z)}{g(z)}$ is regular in $|z| \le R$

and $|g(z)| = 1$ on the boundary, we have

$$\left(\frac{R}{r}\right)^n |f(0)| \le \frac{|f(0)|}{|g(0)|} \le M,$$

which gives the bound above.
Using Stirlings formula

$$\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + \log \sqrt{2\pi} + O\left(\tfrac{1}{z}\right),$$

valid uniformly outside any angle which contains the negative real axis in its interior, we get easily from our earlier estimations of $L(s,\chi)$ and $\xi(s)$ for $\sigma \geq \delta > 0$ that for $|s-2| \leq 2R$, $\xi(s)$ is bounded by $O(R^R)$ and $\xi(s,\chi)$ by $O(q^R R^R)$, so the number of zeros in $|s-2| < R$ is found to be $O(R \log qR)$.

More precise results are obtained by following the variation of the argument of $\xi(s)$ or $\xi(s,\chi)$ around the rectangle with the vertices 2, $2+iT$, $-1+iT$, -1

We shall denote the number of zeros in this rectangle by $N(T)$ for $\xi(s)$ (or $\zeta(s)$) and $N(T,\chi)$ for $\xi(s,\chi)$ (or $L(s,\chi)$), and counting zeros that may lie on the short sides of the rectangle with one half their multiplicity.

We may consider the variation of the argument of $\xi(s)$ or $\xi(s,\chi)$ from $\frac{1}{2}$, 2, $2+iT$, $\frac{1}{2}+iT$, and the upper half of the rectangle will add the same amount by virtue of the functional equation.

a factor

$$\pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \text{ or } \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right)$$

gives in both cases that the argument increases from $\frac{1}{2}$ to $\frac{1}{2}+iT$ by

$$\frac{T}{2}\left(\log \frac{qT}{2\pi} - 1\right) + O(1).$$

Also the variation of the argument of $L(s,\chi)$ or $\zeta(s)$ on $\sigma = 2$ is seen to be bounded since

$$\Re\, L(s,\chi) \geq 1 - \sum_{n=2}^{\infty} \frac{1}{n^2} = 2 - \frac{\pi^2}{6} > \frac{1}{3}.$$

It remains to estimate the variation of the argument of $\xi(s)$ or $L(s,\chi)$ on the stretch $2+iT$, $\frac{1}{2}+iT$ and $\frac{1}{2}$, 2. We assume at first that no zero $\rho = \beta + i\gamma$ has $\gamma = T$ and we look at

$$f(z) = L(2+iT-z,\chi) + \overline{L(2+iT-\bar{z},\chi)},$$

we have $|f(0)| > \frac{2}{3}$ and in the circle $|z| \leq \frac{7}{4}$ we have

$$|f(z)| < c\, q(1+T)).$$

Thus by our earlier lemma, the number of zeros of $f(z)$ in $|z| \leq \frac{3}{2}$ is bounded by

$$n < c' \log q(2+T).$$

But the zeros on the positive real axis are simply the points where $L(2+iT-z,\chi)$ is purely imaginary. Between these the argument of L can vary at most π so the total variation of $L(\sigma+iT,\chi)$ when σ goes from 2 to $\frac{1}{2}$ is $\leq (n+2)\pi$

$$= O(\log q(2+T)).$$

A similar argument on the stretch $(\frac{1}{2}, 2)$ gives $O(\log q)$.

The total variation around the rectangle is thus

$$T\left(\log\frac{qT}{2\pi}-1\right)+O(\log q(2+T))$$

dividing by 2π, we find

$$N(T,\chi)=\frac{T}{2\pi}\left(\log\frac{qT}{2\pi}-1\right)+O(\log q(2+T))$$

which holds whether χ is even or odd, and also in the case $q=1$, for $N(T)$.

Now clear that

$$\xi(s)=c'e^{cs}\prod_\rho\left(1-\frac{z}{\rho}\right)e^{\frac{z}{\rho}},$$

and

$$\xi(s,\chi)=c'_\chi e^{c_\chi s}\prod_\rho\left(1-\frac{z}{\rho}\right)e^{\frac{z}{\rho}},$$

where in each case ρ runs through the zeros of the function on the left-hand side and c' and c constants and the c'_χ and c_χ constants depending on χ only. Product-formulas can also be given for $\xi(s)$ and $L(s,\chi)$. We shall instead look at the logarithmic derivative

$$\frac{\zeta'}{\zeta}(s) = c'' - \frac{1}{s-1} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s-2n} + \frac{1}{2n}\right),$$

and for χ primitive and even

$$\frac{L'}{L}(s,\chi) = c''_{\chi} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{s-2n} + \frac{1}{2n}\right),$$

and χ primitive odd,

$$\frac{L'}{L}(s,\chi) = c''_{\chi} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \sum_{n=0}^{\infty}\left(\frac{1}{s-(2n+1)} + \frac{1}{2n+1}\right).$$

From our results about $N(T,\chi)$ easy to show

$$\sum_{\rho}\left|\frac{1}{s-\rho} + \frac{1}{\rho}\right| = O\left(\log^2(2+|t|)\right)$$

for $s = \sigma + it$; $A > \sigma \geq 1+\delta$.

and $\sum_{n=1}^{\infty}\left|\frac{1}{z-n} + \frac{1}{n}\right| = O\left(\log(2+|t|)\right)$

Let $\alpha > 1$, we have,

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{s+1}}{s(s+1)}\,ds = \begin{cases} 0 & \text{for } 0 \le x \le 1, \\ x-1 & \text{for } x \ge 1. \end{cases}$$

Thus if

$$f(s) = \sum_{1}^{\infty} \frac{c_n}{n^s}$$

is absolute convergent for $\sigma > 1$, then

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{s+1}}{s(s+1)} f(s)\,ds = \sum_{n \le x} (x-n) c_n =$$

$$= \int_0^x \Big(\sum_{n \le t} c_n\Big)\,dt.$$

We define $\Lambda(n) = \begin{cases} \log p & \text{for } n = p^r, r \ge 0 \\ 0 & \text{for others.} \end{cases}$

(von Mangoldt function).

Then

$$\sum_n \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s),$$

and

$$\sum_n \chi(n)\frac{\Lambda(n)}{n^s} = -\frac{L'}{L}(s,\chi).$$

Write

$$\psi(x) = \sum_{n \le x} \Lambda(n),$$

and

$$\psi_\chi(x) = \sum_{n \le x} \chi(n)\Lambda(n),$$

then

$$\int_0^x \psi(t)\,dt = \sum_{n} (x-n)\,\Lambda(n)$$

$$= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'}{\zeta}(s)\right) ds.$$

Here we may use the previous expansion for $-\frac{\zeta'}{\zeta}$ and can integrate term by term since integral exists if we take absolute values everywhere.
We get easily

$$\int_0^x \psi(t)\,dt = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} + O(x),$$

and similarly

$$\int_0^x \psi_\chi(t)\,dt = -\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} + O(x),$$

where the ρ run through the zeros of $\xi(s)$ or $\xi(s,\chi)$ respectively.

To proceed we need to show that neither $\zeta(s)$ or any of the $L(s,\chi)$ have a zero with real part 1.
For $\sigma > 1$ we have

$$R\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{L'}{L}(\sigma+it,\chi) - \frac{L'}{L}(\sigma+2it,\chi^2)\right) \geq 0 .$$

If either $t \neq 0$ or $\chi^2 \neq \chi_0$ we get easily a contradiction if $1+it$ is a zero of $L(s,\chi)$, by letting $\sigma \to 1$ and seeing that the lefthand side of the above equation then would tend to $-\infty$.

In the case $t=0$, $\chi^2 = \chi_0$, we look at $\zeta(s)L(s,\chi)$ and see that

$$\zeta(s)L(s,\chi) = \sum \frac{c_n}{n^s} \quad \text{with } c_n \geq 0, c_{n^2} \geq 1.$$

If $L(s,\chi)$ has a zero at $s=1$ then $\zeta(s)L(s,\chi)$ is an integral function, so its powerseries around $s=2$ converges everywhere. The k'th coefficient is in absolute value

$$\frac{1}{k!}(-1)^k\left(\zeta(s)L(s,\chi)\right)^{(k)} = \frac{1}{k!}\sum \frac{c_n \log^k n}{n^2}$$

$$\geq \frac{1}{k}\sum_n \frac{2^k \log^k n}{n^4},$$

comparing this to the k'th coefficient in the powerseries of $\zeta(2s)$ around $s=2$ we find that to be in absolute value

$$\frac{1}{k!}\sum_n \frac{2^k \log^k n}{n^4},$$

but this powerseries can not converge beyond $s=\frac{1}{2}$ since $\zeta(2s)$ has a pole there. This gives us a contradiction so $L(1,\chi)\neq 0$.

From this we now easily conclude

$$\int_0^x \psi(t)\,dt = \frac{x^2}{2} + o(x^2)$$

and

$$\int_0^x \psi_\chi(t)\,dt = o(x^2).$$

All that is needed is to show that in each case

$$\sum_\rho \frac{x^{\beta+1}}{|\rho(\rho+1)|} = o(x^2)$$

where $\rho=\beta+i\gamma$.

We can always chose a T so large that

$$\sum_{|\gamma|\geq T} \frac{1}{|\rho(\rho+1)|} < \frac{\varepsilon}{2}$$

if ε is a given positive quantity.

Then

$$\sum_{\rho} \frac{x^{\beta+1}}{|\rho(\rho+1)|} < \sum_{|\gamma|<T} \frac{x^{\beta+1}}{|\rho(\rho+1)|} + \frac{\varepsilon}{2} x^2,$$

since in the finite sum $|\gamma| < T$ all exponents $\beta+1 < 2$; this will be $< \frac{\varepsilon}{2} x^2$ for x sufficiently large. So

$$\sum_{\rho} \frac{x^{\beta+1}}{|\rho(\rho+1)|} < \varepsilon x^2$$

for $x > x_0$, which proves the asymptotic relations.

If we write

$$\psi_{q,l}(x) = \sum_{\substack{n \leq x \\ n \equiv l(q)}} \Lambda(n)$$

We see that

$\psi_{q,l}(x)$ differs only slightly from

$$\frac{1}{\varphi(q)} \sum_{q'/q} \sum_{\chi_{q'}} \bar{\chi}(l)\, \psi_\chi(x)$$

(The difference being $\leq \nu(q) \log x$, $\nu(q)$ being no of primefactors of q.)

From this we get easily

$$\int_0^x \psi_{q,l}(t)\,dt = \frac{1}{\varphi(q)} \frac{x^2}{2} + o(x^2)$$

in addition to our previous

$$\int_0^x \psi(t)\,dt = \frac{x^2}{2} + o(x^2)$$

From these relations

$$\psi(x) = x + o(x)$$

and

$$\psi_{q,l}(x) = \frac{1}{\varphi(q)} x + o(x)$$

follow easily, we have

$$\frac{1}{h}\int_{x-h}^{x} \psi(t)\,dt \leq \psi(x) \leq \frac{1}{h}\int_{x}^{x+h} \psi(t)\,dt$$

or

$$x - h - \frac{1}{h} o(x^2) \leq \psi(x) \leq x + h + \frac{1}{h} o(x^2)$$

Take $h = \frac{\varepsilon}{2} x$ and x so large that $\frac{2}{\varepsilon x} o(x^2) < \frac{\varepsilon}{2} x$, and we get

$$x - \varepsilon x \leq \psi(x) \leq x + \varepsilon x$$

for $x > x_0$. Similarly for $\psi_{q,l}(x)$.