

November 16, 1972

Dear Bill,

Thanks for your preprints, which came at a very opportune time. I was about to assume in my Antwerp notes that $p \neq 2$, because I couldn't prove the following facts, which is of course a consequence of "The restriction..."

The support of the character of an absolutely cuspidal representation of $GL(2, F)$, F a non-archimedean local field, does not contain $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ if $|\alpha| \neq |\beta|$.

As an aside, what does the word p -adic mean to you? Can a p -adic field have positive characteristic? I knew of course that this assertion should be a consequence of a Shalika-Mautner type theorem. But my attempts at finding a statement like your Theorem 3 failed.

Just to make sure that I am using your results correctly let me sketch the verification of the above corollary. σ : the representation of $U = Z(F)G(O_F)$ which you denote ϵP . $\Pi = \text{Ind}(\sigma, G(F, u))$. Commuting algebra of Π formed by functions λ such that

$$(*) \quad \lambda(k_1 h k_2) = \sigma(k_2^{-1}) \lambda(h) \sigma(k_1^{-1}).$$

If φ such that $\varphi(ug) = \sigma(u)\varphi(g)$, $u \in U$, then $\lambda : \varphi \rightarrow \psi$ with

$$\psi(h) = \int_{Z(F) \backslash G(F)} \lambda(g) \varphi(gh)$$

Consider a λ satisfying (*) and take

$$h = \begin{pmatrix} a & y \\ 0 & b \end{pmatrix} \quad |a| \leq |b|$$

ϖ is a generator of the maximal ideal of O_F . From (*)

$$\sigma(k^{-1}) \lambda(h) = \lambda(h) \sigma(h k h^{-1})$$

if

$$k \in U \cap h^{-1} U h.$$

In particular if $x \in O_F$

$$\sigma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \lambda(h) = \lambda(h) \sigma \begin{pmatrix} 1 & \frac{ax}{b} \\ 0 & 1 \end{pmatrix}$$

It follows from your analysis that if your c is even then $\lambda(h)$ is 0 unless $|a| = |b|$ and that if c is odd $\lambda(h)$ is 0 unless $|a| = |b|$ or $|\frac{a}{b}| = |\varpi|$. Moreover in the second case, trace $\lambda(h) = 0$ when $|\frac{a}{b}| = |\varpi|$. More generally if h has eigenvalues in F , there is a k in $G(O_F)$ so that

$$k^{-1} h k = \begin{pmatrix} a & y \\ 0 & b \end{pmatrix} = g \quad \left| \frac{a}{b} \right| \leq 1.$$

If $|a| < |b|$

$$\text{trace } \lambda(h) = \text{trace } \sigma(k^{-1}) \lambda(h) \sigma(k) = \text{trace } \lambda(g) = 0.$$

Let $\{g_\alpha\}$ be a set of representations for $U \backslash G(F)$. Let σ act on V and Π on W . If $\{v_\beta\}$ is a basis of V , we may take as a basis for W the functions $\varphi_{\alpha\beta}$, where $\varphi_{\alpha\beta}$ is 0 outside of Ug_α and $\varphi_{\alpha\beta}(ug_\alpha) = \sigma(u)v_\beta$. We take g_α of the form $t_m k$ where

$$t_m = \begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix} \quad m \geq 0$$

and where k runs over a set of representatives for $K \cap t_m^{-1} K t_m \backslash K$. The $\varphi_{\alpha\beta}$ can be used to calculate the trace.

What we have to do to prove the corollary is to show that if

$$h_0 = \begin{pmatrix} a_0 & 0 \\ 0 & b_0 \end{pmatrix} \quad \left| \frac{a_0}{b_0} \right| \neq 1$$

and if t is a function with support in a small neighbourhood of h_0 then

$$\text{trace } \lambda \Pi(f) = 0$$

for any λ in the commuting algebra.

We may suppose that if $f(h) \neq 0$ then h has eigenvalues in F of different absolute values. Let W_α be the functions with support in Kg_α . Let $\lambda \Pi(f) = A$ and let $A = (A_{\alpha\beta})$ where $A_{\alpha\beta} : W_\alpha \rightarrow W_\beta$. We have to show that $\text{trace } A_{\alpha\alpha} = 0$. Now $A : \varphi \rightarrow \psi$ with

$$\psi(x) = \int_{G(F)} \lambda(g) \varphi(gxh) dg$$

$A_{\alpha\alpha} : \varphi \rightarrow \psi$ with

$$\begin{aligned} \psi(g_\alpha) &= \int_{\{g \mid gg_\alpha h = ug_\alpha, u \in U\}} \lambda(g) \varphi(gg_\alpha h) dg \\ &= \int_{\{g \mid gg_\alpha h = ug_\alpha\}} \lambda(g) \sigma(u) \varphi(g_\alpha) dg \end{aligned}$$

Thus

$$A_{\alpha\alpha} = \int_{\{g \mid gg_\alpha h = ug_\alpha\}} \lambda(g) \sigma(u) dg.$$

Since

$$\lambda(g) \sigma(u) = \lambda(u^{-1}g) = \lambda(g_\alpha h^{-1} g_\alpha^{-1})$$

we have

$$\text{trace } A_{\alpha\alpha} = \int \text{trace } \lambda(g_\alpha h^{-1} g_\alpha^{-1}) = 0.$$

Thanks once again for the preprints,
Bob

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