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Dear Bill,

I continue the previous letter. Since I am constantly on the verge of falling into an abyss of categorical confusion and notational disaster, I reiterate some simple facts.

Take G' to be a connected reductive group over F with a Cartan subgroup T' split over F. This data allows one to construct the associated group $\widehat{G}'_0$. Let $\operatorname{Inn} G'(\overline{F})$ be the group of inner automorphisms of G over $\widehat{F}$, let $\operatorname{Aut} G'(\overline{F})$ be the group of all automorphisms of G over $\overline{F}$, and let $\Sigma(G')$ be the quotient. I observe that

$$1 \longrightarrow \operatorname{Inn} G'(\overline{F}) \longrightarrow \operatorname{Aut} G'(\overline{F}) \longrightarrow \Sigma(G') \longrightarrow 1$$

splits.

To split it choose an order on the roots and let Δ be the associated set of simple roots. For each $\alpha \in \Delta$, choose an isomorphism $z \to x_\alpha(z)$ of the additive group with the one-parameter group defined by α. If ω is an automorphism of L, the lattice of roots of T', which takes Δ to itself, [**2**] there is a unique automorphism ω of G satisfying $\omega\lambda(\omega t) = \lambda(t)$, $t \in T'(\overline{F})$ and $\omega\big(x_\alpha(z)\big) = x_{\omega\alpha}(z)$. The set of automorphisms obtained in this way are all defined over F and form a group. Because of standard facts this group splits the exact sequence above.

Because the Killing form $B(\widehat{\lambda}, \widehat{\mu})$, $\widehat{\lambda}$, $\widehat{\mu} \in \widehat{L}$, is given by

$$B(\widehat{\lambda}, \widehat{\mu}) = \sum_\alpha \langle \alpha, \widehat{\lambda} \rangle \langle \alpha, \widehat{\mu} \rangle$$

the contragredient of an automorphism of L mapping Δ to itself is an automorphism of $\widehat{L}$ mapping $\widehat{\Delta}$ to itself. This enables us to identify $\Sigma(G')$ and $\Sigma(\widehat{G}'_0)$.

If G is an arbitrary connected reductive group over F we can introduce $\widehat{G}$ explicitly only after choosing an isomorphism $\psi : G' \to G$ of G with a quasi-split group. As is usual in these considerations, G' plays the role of a base point, and ψ of a path connecting G to it. $\widehat{G}$ is defined by means of the projection δ_σ of $a_\sigma = \psi^{-\sigma}\psi$ on $\Sigma(G')$. By definition $\widehat{G}_0 \simeq \widehat{G}'_0$ and $\Sigma(\widehat{G}_0) \simeq \Sigma(\widehat{G}'_0)$. Once ψ is given we may replace the isomorphism [**3**] by an equality. Two choices of ψ give of course isomorphic groups, but if we are to consider $\Pi(G)$ we need the relation between $\widehat{G}$ and G to be more explicit.

Suppose $g \to \alpha(g)$ is an automorphism of G defined over F. Let β be the image of $\psi^{-1}\alpha\psi$ in $\Sigma(G')$. Let γ be an automorphism of G' over F so that $\psi^{-1}\alpha\psi = \gamma\delta$ with δ inner. Then

$$\alpha\psi = \psi\gamma\delta$$

and

$$\alpha\psi^\sigma = \psi^\sigma\gamma\delta^\sigma.$$

Thus

$$\psi^{-1}\psi^\sigma = \delta^{-1}\gamma^{-1}\psi^{-1}\psi^\sigma\gamma\delta$$

so $\delta_\sigma^{-1} = \beta^{-1}\delta^{-1}\sigma\beta$ or $\beta\delta_\sigma = \delta_\sigma\beta$.

β may also be regarded as an element of $\Sigma(\widehat{G}_0) \simeq \Sigma(\widehat{G}'_0)$. We then use the splitting $\Sigma(\widehat{G}'_0) \hookrightarrow \mathrm{Aut}(\widehat{G}'_0)$ used to define $\widehat{G}$ to regard β also as an automorphism of $\widehat{G}_0$. It commutes with the action of $\mathfrak{G}(K/F)$ and $\beta : g \times \sigma \to \beta(g) \times \sigma$ is an automorphism of $\widehat{G}$.

[**4**] The set $\Phi(G)$ of the last letter can be defined for any local or global field; and to any $\{\varphi\} \in \Phi(G)$ I expect that it will eventually be possible to associate a finite set $\Pi_{\{\varphi\}}(G)$ in $\Pi(G)$. Over a global field $\Pi(G)$ is this set of classes occurring in the space of automorphic forms. Of course unless F is $\mathbf{R}$ or $\mathbf{C}$ one does not expect the union of these $\Pi_{\{\varphi\}}(G)$ to be all of $\Pi(G)$. It is not even clear that one should expect them to be disjoint when F is global. This at least is the impression given by the notes on "Abelian algebraic groups".

In general the element β above acts on $\Phi(G)$ by $\{\varphi\} \to \{\beta\varphi\}$ where

$$\beta\varphi : w \to \beta\big(\varphi(w)\big).$$

It does seem reasonable to expect that

$$\Pi_{\{\beta\varphi\}}(G) = \Big\{\, g \to \pi\big(\alpha^{-1}(g)\big) \;\Big|\; \pi \in \Pi_{\{\varphi\}}(G) \,\Big\}. \tag{1}$$

This is so when G is abelian or $F = \mathbf{R}$ or $\mathbf{C}$ and $\varphi(W_{\mathbf{C}/\mathbf{R}})$ is contained in no proper parabolic subgroup. These are the only cases so far in which the sets have been defined.

[**5**] For abelian groups one gets by with a formal argument. β acts also on $\widehat{L}$. Since its action commutes with that of $\mathfrak{G}(K/F)$ it also acts on $H_1(W_{K/F}, \widehat{L})$. All one has to do is check that upon transporting β via $H_1(W_{K/F}, \widehat{L}) \xrightarrow{\sim} G(F)$ we obtain α. Since this isomorphism is given by

$$H_1(W_{K/F}, \widehat{L}) \to H_1(C_K, \widehat{L}) \simeq G(K)$$

and since β clearly commutes with restriction we may suppose $F = K$. If $z \to \widehat{\lambda}(z)$ is a 1-cycle of C_K with values in $\widehat{L}$ it corresponds to t in $G(K)$ defined by

$$\lambda(t) = \prod_z z^{\left\langle \lambda, \widehat{\lambda}(z) \right\rangle}$$

Applying β and then the isomorphism we obtain t' where

$$\lambda(t') = \prod_z z^{\left\langle \lambda, \beta\widehat{\lambda}(z) \right\rangle} = \prod_z z^{\left\langle \beta^{-1}\lambda, \widehat{\lambda}(z) \right\rangle} = \beta^{-1}\lambda(t) = \lambda\big(\alpha(t)\big).$$

For $F = \mathbf{R}$ we take G reductive and revert to the notation of the previous letter. T is a compact Cartan subgroup in G and $\widehat{T}$ a standard torus in $\widehat{G}_0$. Composing α with an inner automorphism if necessary we [**6**] may suppose that it leaves T invariant. β acts on $\widehat{T}$, $\widehat{L}$, and L. We may also identify L with the lattice of characters of T even though this identification is not unique. There is an element γ of the normalizer of T in $G(\mathbf{C})$ so that

$$\lambda\big(\alpha(t)\big) = \gamma\beta\lambda(t) \qquad t \in T(\mathbf{R}).$$

We may also suppose $\varphi : \mathbf{C}^\times \to \widehat{T}$. As in the previous letter

$$\widehat{\lambda}\varphi(z) = z^{\langle \mu, \widehat{\lambda} \rangle} \overline{z}^{\langle \nu, \widehat{\lambda} \rangle}.$$

If $\varphi' = \beta\varphi$ then

$$\lambda\varphi'(z) = z^{\langle \beta\mu, \widehat{\lambda} \rangle} \overline{z}^{\langle \beta\nu, \widehat{\lambda} \rangle}.$$

If $\pi_{\tau\mu_1}$ is defined as on p. 25 of the previous letter then

$$g \to \pi_{\tau\mu_1}\big(\alpha^{-1}(g)\big)$$

is the representation $\pi_{\tau\gamma\beta\mu_1} = \pi_{\tau\gamma\mu_1'}$ if μ_1' is defined in terms of $\mu' = \beta\mu$ the way μ_1 was defined in terms of μ. We have also to check that if χ and χ' are constructed as on p. 26 of the previous letter, χ for φ and χ' for φ', then

$$\chi'(c) = \chi\big(\alpha^{-1}(c)\big) \qquad c \in C(\mathbf{R}).$$

I observe that since α is an automorphism of T over $\mathbf{R}$ it does take C to itself. In any **[7]** case the desired relation is a consequence of the previous result for abelian groups.

What I want to do now is to take F to be a local field, assume that $\Pi_{\{\varphi\}}(G)$ has been defined when φ is such that $\varphi(W_{K/F})$ is contained a no proper parabolic subgroup of $\widehat{G}$, and then show how these sets may be defined in general. Note that when F is non-archimedean one really wants to take a direct limit over K. I assume that the sets $\Pi_{\{\varphi\}}(G)$ are disjoint and finite, that each is nonempty and consists of discrete classes, and that (1) is satisfied. It will also be necessary to assume the simple relations (3) and (4) below.

Suppose $\{\varphi\} \in \Phi(G)$. Let $\widehat{P}$ be minimal among the parabolic groups containing $\varphi(W_{K/F})$. Let $\widehat{N}$ be the unipotent radical of $\widehat{P}_0$, the connected component of $\widehat{P}$, and let $\widehat{M}$ be $\widehat{P}/\widehat{N}$. By assumption $\{\widehat{P}\} \in \mathfrak{p}(G)$. Let P be a corresponding parabolic F-subgroup of G, let N be its unipotent radical, and let $M = P/N$. I claim that $\widehat{M}$ is the associated group to M.

This particular point might as well be settled for a general field F. **[8]** We may factor P as MN and regard M as an F-subgroup of G. Let T be a Cartan subgroup of M defined over F. It is also a Cartan subgroup of G. Thus L and $\widehat{L}$ are the same for M and G. Φ_M, the set of roots for M, is a subset of Φ_G. If $\alpha \in \Phi_M$ then $\widehat{\alpha}$ is defined with respect to G by $\beta(H_\alpha) = \langle \beta, \widehat{\alpha} \rangle$, where H_α has its usual significance. Thus $\widehat{\alpha}$ is the same whether it is defined with respect to M or to G (This argument is not quite correct in positive characteristics). It follows immediately that $\widehat{M}_0$, the connected component of $\widehat{M}$, is the connected component of the group associated to M. $\widehat{M}$ is a semi-direct product of $\widehat{M}_0$ and $\mathfrak{G}(K/F)$ so we need only verify that the action of $\mathfrak{G}(K/F)$ on $\widehat{T}$, or, what is the same, in $\widehat{L}$ is correct. Let G' be split over F and let $\psi : G' \xrightarrow{\sim} G$. Modifying ψ by an inner automorphism we may suppose that $\psi : T' \xrightarrow{\sim} T$ and $P' \xrightarrow{\sim} P$ where T' and P' are defined over F. Set $a_\sigma = \psi^{-\sigma}\psi$. In so far as its effect on T' is concerned, a_σ is a product $b_\sigma c_\sigma$ where c_σ is obtained from an inner automorphism normalizing T' and b_σ from an outer automorphism normalizing T' and taking positive roots to positive roots. The action **[9]** of σ on L is contragradient to b_σ. To show that this also gives the action with respect to M we need to show that c_σ comes from an inner automorphism of M' normalizing T'. Since a_σ normalizes P' there is an inner automorphism d_σ from M' so that $b_\sigma c_\sigma d_\sigma$ takes positive roots to positive roots. Then $c_\sigma d_\sigma$ acts trivially on T' and $d_\sigma = c_\sigma^{-1}$. This does it.

I observe that once $\widehat{P}$ and P are chosen then, since ψ is fixed once and for all, not only do we have an isomorphism between $\widehat{M}$ and the associated group to M but a class of isomorphisms given up to inner automorphisms by an element of $\widehat{M}_0$. Modifications by such inner automorphisms are of no import. According to Mostow (Fully reducible subgroups of algebraic groups, Amer. Jour. 1956) the Zariski closure of $\varphi(W_{K/F})$ is contained in a maximal fully reducible subgroup of $\widehat{P}$, which we may identify with $\widehat{M}$. Moreover by assumption $\varphi(W_{K/F})$ is contained in no proper parabolic subgroup of $\widehat{M}$.

The induction assumption and our choice of $\widehat{M}$, M, $\widehat{P}$, P yield therefore a finite subset of $\Pi(M)$. We denote it by $\Pi_{\{\varphi\}}(M; P, \widehat{P})$. It does not depend **[10]** on $\widehat{M}$. We must know however how it depends on $\widehat{P}$.

Suppose $\varphi(W_{K/F})$ is contained minimally in both $\widehat{P}$ and $\widehat{P}'$. Then

$$(\widehat{P} \cap \widehat{P}')\widehat{N} \supseteq (\widehat{P}_0 \cap \widehat{P}'_0)\widehat{N}$$

and the right side is a parabolic subgroup of $\widehat{G}_0$ (Borel-Tits). The left side contains $\varphi(W_{K/F})$ which projects into $\mathfrak{G}(K/F)$. Thus it is a parabolic subgroup of $\widehat{G}$. Since it is contained in $\widehat{P}$ on one hand and contains $\varphi(W_{K/F})$ on the other, it is equal to $\widehat{P}$. Thus (4.4b again) $\widehat{P}'_0$ contains a maximal reductive subgroup of $\widehat{P}_0$. For the same reason $\widehat{P}_0$ contains a maximal reductive subgroup of $\widehat{P}'_0$. Take a Cartan subgroup $\widehat{T}$ contained in $\widehat{P}_0 \cap \widehat{P}'_0$. We see that if both root vectors $X_{\widehat{\alpha}}$, $X_{-\widehat{\alpha}}$ lie in the Lie algebra $\widehat{p}$ of $\widehat{P}$ then they must also lie in $\widehat{p}'$. Thus $\widehat{P}_0$ and $\widehat{P}'_0$ have a common maximal reductive subgroup. Take a maximal reductive subgroup of $\widehat{P} \cap \widehat{P}'$ containing $\varphi(W_{K/F})$. Since, by Mostow, any two maximal reductive subgroups of $\widehat{P} \cap \widehat{P}'$ are conjugate it must contain maximal reductive subgroups of $\widehat{P}_0$ and of $\widehat{P}'_0$. Since it projects onto $\mathfrak{G}(K/F)$ it must be a maximal reductive subgroup **[11]** of both $\widehat{P}$ and $\widehat{P}'$. We may take it as $\widehat{M}$.

Thus for our purposes we may fix $\widehat{M}$ and let $\widehat{P}$ vary over the parabolic subgroups containing it. In order to simplify the notation and the explanations I replace φ and $\widehat{M}$ by conjugates and suppose that $\widehat{M}$ contains the standard Cartan subalgebra $\widehat{T}$. Let A be the space of vectors in $L \otimes \mathbf{R}$ invariant under the action of $\mathfrak{G}(K/F)$ and orthogonal to the roots of $\widehat{M}$. There is a bijective correspondence between the set of connected components of the complement in A of the hyperplanes $\{\, a \in A \mid \langle a, \widehat{\alpha} \rangle = 0 \,\}$, where $\widehat{\alpha}$ is a root of $\widehat{T}$ in $\widehat{G}_0$ but not in $\widehat{M}_0$, and the set of parabolic subgroups containing $\widehat{M}$. If $\widehat{P}$ is such a subgroup we associate to it

$$\{\, a \mid \langle a, \widehat{\alpha} \rangle > 0 \text{ for all } \widehat{\alpha} \text{ such that } X_{\widehat{\alpha}} \in \widehat{p},\ X_{\widehat{\alpha}} \notin \widehat{m} \,\}$$

Conversely if a component W is given, the corresponding $\widehat{P}$ is generated by $\widehat{T}$ and the groups $z \to x_{\widehat{\alpha}}(z)$ where $\widehat{\alpha}$ lies in

$$\{\, \widehat{\alpha} \mid \langle a, \widehat{\alpha} \rangle \geqslant 0 \text{ for all } a \in W \,\}.$$

[12] Suppose $\widehat{P}$ and $\widehat{P}'$ both contain $\widehat{M}$ and are in addition conjugate under $\widehat{G}_0$. Then there exists ω in $N(\widehat{T})$ so that $\omega(\widehat{P}) = \widehat{P}'$. This ω necessarily normalizes $\widehat{M}$. Moreover $\omega^{-1}\delta_\sigma\omega\delta_\sigma^{-1}$ takes $\widehat{P}$ to itself, normalizes $\widehat{M}$, and is obtained by restricting an automorphism $h \to ghg^{-1}$, $g \in \widehat{G}_0$ to $\widehat{M}$. g must lie in $\widehat{M}_0$ so that $\omega^{-1}\delta_\sigma\omega\delta_\sigma^{-1}$ is an inner automorphism of $\widehat{M}$ determined by an element of $\widehat{M}_0$. Multiplying ω by an inner automorphism from $\widehat{M}_0$ if necessary we may suppose it takes positive roots of $\widehat{M}_0$ to positive roots. Then the image of $\omega^{-1}\delta_\sigma\omega\delta_\sigma$ in the Weyl group is trivial. I shall show later that this implies that, after multiplication by an element of $\widehat{T}$, ω may be supposed to commute with each δ_σ.

Then

$$m \to \omega m \omega^{-1}$$

is an automorphism of the type envisaged in (1). We shall check that there is an element w of $G(F)$ normalizing M so that the automorphism $m \to wmw^{-1}$ and $\widehat{m} \to \omega\widehat{m}\omega^{-1}$ of

M and $\widehat{M}_0$ respectively have the same image **[13]** in $\Sigma(M) = \Sigma(\widehat{M}_0)$. The assumption of (1) allows us to conclude that

$$\Pi_{\{\varphi\}}(M, P, \widehat{P}') = \Big\{ m \to \pi(w^{-1}mw) \ \Big|\ \pi \in \Pi_{\{\varphi\}}(M, P, \widehat{P}) \Big\}. \tag{2}$$

Let D be the lattice of characters of M. Choosing a Cartan subgroup T over F which is contained in M and identifying L with its lattice of rational characters we imbed $D \to L$. If $\delta \in D \subseteq L$ and δ^σ denotes the result of applying $\sigma \in \mathfrak{G}(K/F)$ to it, then $\delta^\sigma(t^\sigma) = \big(\delta(t)\big)^\sigma$. This relation is not a matter of definition. It results because the action of σ on L differs from its Galois action, obtained by regarding L as the lattice of rational characters of T, by the action of an element of the normalizer of T in M. A is equal to the set of invariant elements in $D \otimes \mathbf{R}$. If $B(\cdot,\cdot)$ is the Killing form there is a bijective correspondence between parabolic F-subgroups containing M and connected components in A of the complement of the hyperplanes $B(x,\alpha) = 0$ where α runs over the roots of G which are not roots of M. These are the same connected components met previously because $\frac{2B(x,\alpha)}{B(\alpha,\alpha)} = \langle x, \widehat{\alpha}\rangle$. P corresponds to **[14]**

$$\big\{ x \ \big|\ B(x,\alpha) > 0 \text{ if } \alpha \text{ is a root of } D \text{ but not of } M \big\}.$$

If two components can be taken into each other by an element of the Weyl group normalizing M then the corresponding parabolic groups are conjugate over $\overline{F}$ and hence over F (Borel-Tits). This establishes the existence of the w used previously.

$\widehat{P}$ determines P and M up to conjugacy. Suppose $P' = hPh^{-1}$, $M' = hMh^{-1}$. Then

$$\Pi_{\{\varphi\}}(M', P', \widehat{P}) = \Big\{ m \to \pi(h^{-1}mh) \ \Big|\ \pi \in \Pi_{\{\varphi\}}(M, P, \widehat{P}) \Big\}.$$

This relation is in particular true if $M = M'$ so that indeterminacy is the same as in (2) above.

The above identification of L with the lattice of rational characters of M (which is not exactly cricket) and the assumption that $\widehat{M}$ contains $\widehat{T}$ fixes a P and a $\widehat{P}$ in corresponding classes. If $\widehat{P}'$ is another parabolic subgroup containing $\widehat{M}$ and P' a parabolic F-subgroup containing M and of these two are associated to the same connected component (or chamber) then **[15]** P' and $\widehat{P}'$ are in corresponding classes.

Then the isomorphism between $\widehat{M}$ and the associated group to M determined by the pair P', $\widehat{P}'$ is the same as that determined by P, $\widehat{P}$. I did not stress adequately before that it was the pair P, $\widehat{P}$ which determines this isomorphism, up to conjugation by an element of $\widehat{M}_0$, because I was not sufficiently aware of it. To prove the required statement I review the construction. We are given a split group G_1 over F and $\psi : G_1 \to G$. We have moreover fixed a Cartan subgroup T_1 of G_1 and an order on its positive roots, and thus a collection of standard parabolic subgroups. Modifying ψ by an inner automorphism of G_1 we suppose that $\psi^{-1}(P) = P_1$ is standard and that $\psi^{-1}(M) = M_1$ contains T_1. $\widehat{G}_{10}$, which is isomorphic to $\widehat{G}_0$, is by construction also provided with a set of standard parabolic subgroups. Choose $h \in \widehat{G}_0$ so that $h\widehat{P}h^{-1}$ is standard and so that $h\widehat{M}h^{-1}$ contains $\widehat{T}$. What we showed earlier was that $\widehat{M}_{10}$ could be regarded as a subgroup of $\widehat{G}_{10}$ and that $\widehat{M}_{10} \times \mathfrak{G}(K/F)$ was the associated group to M, defined by means of **[16]** $\psi : M_1 \to M$ together with the order on the roots of M_1 defined by that in the roots of T_1. Since $h\widehat{M}h^{-1} = \widehat{M}_{10} \times \mathfrak{G}(K/F)$ the required isomorphism is constructed.

We have still to prove the assertion which was the cause of this reiteration. We may clearly replace $\widehat{M}$, $\widehat{P}$, and $\widehat{P}'$ by $h\widehat{M}h^{-1}$, $h\widehat{P}h^{-1}$, $h\widehat{P}'h^{-1}$ and hence suppose that $\widehat{P}$ is standard and that $\widehat{M}$ contains $\widehat{T}$. We shall show later that there is an element w in the normalizer of $\widehat{T}$ in $\widehat{G}_0$ which commutes with $\mathfrak{G}(K/F)$ so that $\widehat{w}\widehat{P}'\widehat{w}^{-1} = \widehat{P}''$ is standard and so that $\widehat{w}$ takes positive roots of $\widehat{M}_0$ to positive roots of $\widehat{M}_0'' = \widehat{w}\widehat{M}_0\widehat{w}^{-1}$. Let w be an element in the normalizer of T_1 so that w and $\widehat{w}$ define corresponding characters in the Weyl groups of T and $\widehat{T}$. Then $\psi \circ \operatorname{Ad} w$ takes P_1'' to P_1' where P_1'' is standard. Let $\psi \circ \operatorname{Ad} w$ take M_1'' to M. ψ and $\psi \circ \operatorname{Ad} w$ give two different identifications λ, $\lambda'' : L_{G_1} \to L_M$. $\lambda'' = \lambda \circ w$, where w here denotes the contragredient effect of w on L_{G_1}.

If we build the associated group with respect to the first identification we get $\widehat{M}$, with respect to the second we get $\widehat{M}'' = w\widehat{M}w^{-1}$. The resulting **[17]** identification of these two groups themselves is given by taking $\widehat{T}$ to $\widehat{T}''$ in such a way that if $\widehat{t} \to \widehat{t}''$ then $\widehat{w}\widehat{\mu}(\widehat{t}) = \widehat{\mu}(\widehat{t}'')$ for each weight $\widehat{\mu}$. Moreover $x_{\widehat{\alpha}}(z) \to x_{\widehat{w}\widehat{\alpha}}(z)$ if $\widehat{\alpha}$ is a simple positive root of $\widehat{M}$. Observe that $\widehat{T}$ is the standard Cartan subgroup of both $\widehat{M}$ and $\widehat{M}''$. Finally $\mathfrak{G}(K/F) \to \mathfrak{G}(K/F)$ is the identity. Consider the map $\widehat{m} \to \widehat{s}\widehat{w}\widehat{m}\widehat{w}^{-1}\widehat{s}^{-1}$ where $\widehat{s}$ is an element of $\widehat{T}$ yet to be chosen. It has the right effect on $\widehat{T}$. Moreover if $\widehat{s}$ is chosen to be invariant under $\mathfrak{G}(K/F)$ it has the right effect on $\mathfrak{G}(K/F)$. There are $c_\alpha \in K^\times$ so that $\widehat{w}x_{\widehat{\alpha}}(z)\widehat{w}^{-1} = x_{\widehat{w}\widehat{\alpha}}(c_\alpha z)$. We choose H in the Lie algebra of $\widehat{T}$ so that $\widehat{\alpha}(\exp H) = c_\alpha$. It is clear that $c_{\sigma\alpha} = c_\alpha$ if $\sigma \in \mathfrak{G}(K/F)$. Replace H by

$$H' = \frac{1}{\big[\mathfrak{G}(K/F) : 1\big]} \sum_\sigma \sigma(H)$$

and assume it is invariant under $\mathfrak{G}(K/F)$ and take $\widehat{s} = \exp(H)$. Replacing $\widehat{w}$ by $\widehat{s}\widehat{w}$ we see that the identification is given by conjugation by $\widehat{w}$. This is what was to be proved.

I have now to define $\Pi_{\{\varphi\}}(G)$. The construction will be such that replacing **[18]** M and P by gHg^{-1}, gPg^{-1}, $g \in G(F)$, and $\Pi_{\{\varphi\}}(M, P, \widehat{P})$ by

$$\Big\{\, m \to \pi(g^{-1}mg) \;\Big|\; \pi \in \Pi_{\{\varphi\}}(M, P, \widehat{P}) \,\Big\}$$

does not change the result, which will therefore at least be independent of M. It will also be clearly independent of the choice of φ in $\{\varphi\}$.

Let $\widehat{B}$ be the lattice of characters of $\widehat{M}$. $\widehat{B}$ is the orthogonal complement of the collection of roots α and $B = \operatorname{Hom}(\widehat{B}, \mathbf{Z})$ is a quotient space of L which my be regarded as the lattice of rational characters of the connected component of the centre Z of H. The map $\widehat{B} \to \widehat{L}$ yields $\widehat{G} \to \widehat{Z}$ and every element $\{\varphi\}$ of $\Phi(G)$ determines ψ in $\Phi(Z)$. We assume that, for the representations whose existence we are taking for granted now, we have

$$\pi(z) = \epsilon_\psi(z)I \qquad z \in Z(F) \tag{3}$$

where ϵ_ψ is the quasi-character of $Z(F)$ associated to ψ. This relation is of course easily verified for the real field.

In fact, if we use the notation of the previous letter we have $1 \to C \to Z$ and $\widehat{N}' \leftarrow \widehat{B} \leftarrow 0$. One sees easily that $Z(\mathbf{R}) = Z^0(\mathbf{R})C(\mathbf{R})$. **[19]** On $C(\mathbf{R})$ we are alright by construction. On $Z^0(\mathbf{R})$ the discussion on pp. 27–28 of the previous letter gives what is necessary.

In general

$$\Big|\widehat{\lambda}\big(\varphi(x)\big)\Big| = |x|^{\langle \mu, \widehat{\lambda}\rangle} \qquad \widehat{\lambda} \in \widehat{B},\ x \in K^\times$$

where μ is a certain invariant element in $B \otimes \mathbf{R}$, that is, an element of A. On the other hand every element z of $Z(F)$ defines an element $\widehat{\mu}(z)$ of $\widehat{B} \otimes \mathbf{R}$ by

$$\left|\lambda(z)\right| = \begin{cases} |\varpi_F|^{-\left\langle \lambda, \widehat{\mu}(z)\right\rangle} & F \text{ non-archimedean} \\ e^{\left\langle \lambda, \widehat{\mu}(z)\right\rangle} & \text{archimedean.} \end{cases}$$

By construction

$$\left|\epsilon_\psi(z)\right| = \begin{cases} |\varpi_F|^{-\left\langle \mu, \widehat{\mu}(z)\right\rangle} \\ e^{\left\langle \mu, \widehat{\mu}(z)\right\rangle} \end{cases}.$$

We shall consider why P and $\widehat{P}$ such that μ belongs to the closure of the chamber determined by P. Observe two things. This is sensitive to the choice of $\widehat{P}$, because that affects the identification of $\widehat{M}$ with the associated groups to M. It can always be arranged because without affecting the identification we can let P vary over all parabolic groups containing M.

There is one thing further to assume. For those classes in $\Phi(M)$ from which we start, namely those for which $\varphi(W_{K/F})$ is contained in no proper **[20]** parabolic subgroup, the representations in $\Pi_{\{\varphi\}}(M)$ are almost unitary, that is each $\pi(m)$ is a scalar multiple of a unitary operator

$$\pi^*(m)\pi(m) = a(m)I \qquad a(m) \in \mathbf{R}.$$

Suppose first that $\mu = 0$. Then I take any $\pi_0 \in \Pi_{\{\varphi\}}(M, P, \widehat{P})$ and consider

$$\Pi = \operatorname{Ind}(G, MN, \delta_p\pi_0)$$

where

$$\delta_p\pi_0(m \cdot n) = \delta_p(m)\pi_0(m)$$

and δ_p is the usual scalar function. Π is almost unitary. $\Pi_{\{\varphi\}}(G)$ will consist of all those irreducible π which occur as subrepresentations of this Π for some choice of π_0. That this collection is well-defined should follow from the theory of intertwining operators (Harish-Chandra, *On the theory of the Eisenstein integral,* Theorem 6, and *Harmonic analysis on reductive p-adic groups* §11).

When $\mu = 0$ any choice of P is permissible. Now consider the general case. We introduce the representation Π again. It is realized in the space V of all continuous functions ψ on $G(\mathbf{R})$ with values in the space on which π_0 acts satisfying **[21]**

$$\psi(nmg) = \delta_p(nm)\pi(m)\psi(g)$$

Let ${}^*P$ be the parabolic group defined by

$${}^*\theta = \left\{\, \alpha \;\middle|\; B(\mu, \alpha) = 0 \,\right\}$$

${}^*P$ contains P. Let ${}^*\overline{N}$ be the unipotent radical of the opposite group. I claim that for each ψ in V and each g in $G(\mathbf{R})$ the integral

$$\psi'(g) = \int_{{}^*\overline{N}} \psi(ng)\, dn$$

is well-defined. Let W be the space of all ψ for which ψ' vanishes identically. W is invariant and the representation on the quotient space has a finite composition series (closed subspaces). $\Pi_{\{\varphi\}}(G)$ will consist of all constituents of the representation on V/W for any and all choices of π_0. It should not be too hard to show that $\Pi_{\{\varphi\}}(G)$ is well-defined.

I leave this for now, as well as the verification that the sets $\Pi_{\{\varphi\}}(G)$ are disjoint and that when F is archimedean their union exhausts $\Pi(G)$. You are probably not terribly interested. I will **[22]** talk these things over with Mrs. Shelstad, who by the way has answered some of the questions at the end of my last letter, and only write if the proofs don't work out.

There are a few other simple things to be done. If $F = \mathbf{C}$ or $G = \mathrm{GL}(n)$ the sets $\Pi_{\{\varphi\}}(G)$ should consist of a single element. For $\mathbf{C}$ this should follow immediately from the work of Zhelobenko. It would also be good to take the work of Hirai, which suggested the above definition, and of Wallach and Johnson into account.

Of course for applications to the global theory much more must be done. In particular the structure of the individual sets $\Pi_{\{\varphi\}}(G)$ has to be more closely investigated.

All the best,
Bob

Compiled on November 12, 2024.