Late June 1974

Dear Bill,

I have been ruminating further along the lines of our discussion and I now believe I can analyze the formal aspects of the situation and reduce everything to three specific representation-theoretic problems. Since we are leaving for Montreal today I don't have time to describe the analysis; that I shall postpone to our return. However let me pose the two problems to you now to spur you into solving them. I pose the first for groups quasi-split and split over an unramified extension. You may prefer, at the moment, to treat it only for Chevalley groups.

1. *Does every irreducible factor of a unitary unramified principal series contain the trivial representation of some special maximal compact?*

Of this problem you are of course already aware. It means that the group

$$C = \widehat{L}(T^0_{\mathrm{ad}})/\operatorname{Im}\widehat{L}(T^0)$$

acts transitively on each Π_φ. Suppose χ is the character of C trivial on the subgroup C_0 of C acting trivially on Π_φ. Choose a special maximal compact K^0 and hence $\pi^0 \in \Pi_\varphi$. If $\zeta_1, \dots, \zeta_r$ are the values taken by χ set

$$\pi^i = \sum_{\substack{c \in C_0 \backslash C \\ \chi(c) = \zeta_i}} c\pi^0$$

so that

$$(*) \qquad \pi_\varphi = \bigoplus \pi^i.$$

Suppose M is a Levi factor of a *PSG* of G over F. Let S^0 be T^0 regarded as a *CSG* of M. We have

$$\begin{array}{ccc}
\widehat{L}(T^0) & \longrightarrow & L^1(T^0_{\mathrm{ad}}) \\
\Big\vert \wr & & \Big\downarrow \text{(surjective)} \\
\widehat{L}(S^0) & \longrightarrow & \widehat{L}(S^0_{\mathrm{ad}})
\end{array}$$

If χ a character of $\widehat{L}(T^0_{\mathrm{ad}})$, can be obtained by pulling back a character of $\widehat{L}(S^0_{\mathrm{ad}})$ and if τ_φ is the principal series of M corresponding to φ so that π_φ is obtained from τ_φ buy a normalized induction, then the decomposition $(*)$ as a consequence of a corresponding decomposition

$$\tau_\varphi = \bigoplus \tau^i.$$

I shall try to convince you in a later letter that, given χ, one can choose M so that M_{ad} is isomorphic over F to a product of groups of the form

$$\operatorname{Res}_{K/F} \operatorname{PSL}(m)$$

with K/F unramified.

These comments may be a help in solving the first problem. They also form an introduction to the second.

Take n unramified extensions K_i $1 \leqslant i \leqslant n$, of F and take unramified extensions E_i/K_i of degrees m_i. Choose the basis of O_{E_i} over O_{K_i} (also one of E_i) and use it to imbed $E_i^\times$ in $\mathrm{GL}(m_i, K_i)$. Let G be a closed subgroup of $\prod_i \mathrm{GL}(m_i, K_i)$ containing

$$\left\{ g \;\middle|\; \eta(g) \in \prod K_i^{m_i} \right\}$$

Here

$$\eta : g \to \prod \det g_i \in \prod K_i^{m_i}.$$

Let χ be a character of $\prod K_i^\times$ trivial on $\eta(G)$ and such that the kernel of χ in $K_i^\times$ is $\mathrm{Nm}\, E_i^\times$

Let π be a unitary unramified principal series representation of G and let Π be the set of irreducible components of its restrictions to G.

$$H = \prod K_i^\times / \eta(G)$$

acts on Π. Let its kernel be H^0.

2. *If $m = 1$ and $G = K^\times \mathrm{SL}(m, K)$ then the inverse image of H^0 in $K^\times$ is $\{\alpha \mid m|o(\alpha)\}$ if and only if π is a principal series representation corresponding to a character*

$$\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_m \end{pmatrix} \to \nu(\alpha_1, \dots, \alpha_m) \zeta^{o(\alpha_2) + 2o(\alpha_2) + \cdots + (m-1)o(\alpha_m)}$$

Here $o(\alpha)$ is the order of α.

Anyhow suppose H^0 is contained in the kernel of χ. Let $\pi^0 \in \Pi$ be the representation containing the trivial representation of G in $\prod_i \mathrm{GL}(m, O_{K_i})$.

Form

$$\Theta = \sum_{H^0 \backslash H} \chi(h) \Theta_{h\pi^0}.$$

Let T be the set of all $g = \prod g_i$ in G with $g_i \in E_i^\times$. It is clear that Θ, which one has to prove is a function (this is known only in characteristic 0), has support in $\bigcup_{g \in \prod \mathrm{GL}(m_i, K_i)} g^{-1} T g$.

It is clear that

$$\Theta(\gamma^w) = \Theta(w^{-1} \gamma w) = \chi\big(\eta(w)\big)^{-1} \Theta(\gamma)$$

3. *Find a formula for $\Theta(\gamma)$ when $\gamma \in T$ is regular.*

A *suggestion*

Let

$$\gamma = (\gamma_1, \dots, \gamma_r)$$

and fix

$$\gamma^0 = (\gamma_1^0, \dots, \gamma_r^0) \qquad E_i = K_i[\gamma_i^0].$$

For each i, χ defines a character χ_i of $K_i^\times$ and of $\mathfrak{G}(E_i/K_i)$

$$\delta_j = \frac{\sum_{\tau(\mathfrak{G}(E_i/K_i))} \chi_i(\tau)\tau(\gamma_i)}{\sum \chi_i(\tau)\tau(\gamma_i^0)} \in K_i^\times$$

if γ is regular. We may introduce

$$\prod \chi_i(\delta_i).$$

Then

$$\Theta(\gamma) = c \prod \chi_i(\delta_i)$$

where c is a constant involving orders of Weyl groups and Gaussian sums for the characters χ_i.

I hope to hear from you soon,
Bob

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