December, 1974

Dear Deligne:

This letter contains the promised proof of the identity connecting the zeta-function of a Shimura variety and what I have called Artin-Hecke L-functions, namely the L-functions associated to automorphic forms. I can of course only carry out the proof for the multiplicative group of a totally indefinite quaternion algebra over a totally real field. What I shall do in fact is carry out the analytic part of the proof for the multiplicative group of any quaternion algebra over a totally real field assuming that my conjectures about the structure of the set of geometric points over finite fields are valid. For a proof of these conjectures in the case of a totally indefinite algebra I rely for the moment on my letter to Rapoport, admittedly a rather shaky support. For calculations which are not worth repeating I shall rely on my Antwerp report and on my letter to you of October, 1973.

I begin by recalling the general situation.

- $G/\mathbf{Q}$ is given. Take it to be connected.
- $h : S \to G$ over $\mathbf{R}$.
-

$$\begin{array}{ccccccc} z & & \mathrm{GL}(1) & & & \searrow^{h_0} & \\ \downarrow & & \downarrow & & & & \text{over } \mathbf{C} \\ (z,1) & & \mathrm{GL}(1)\times\mathrm{GL}(1) & \xrightarrow{\sim} & S & \xrightarrow[h]{} & G \end{array}$$

- My h is the inverse of yours.
- The associate group in Galois form

$$\check{G} = \check{G}^0 \times \mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q}).$$

In $\check{G}^0$ have Borel $\check{B}^0$ and a Cartan subgroup $\check{T}$ of $\check{B}^0$, both normalized by $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$. h_0 yields an orbit under the Weyl group in the cocharacters of a CSG of G and hence an orbit $\{\check{\mu}\} = \{h_0\}$ in the set of characters of $\check{T}$ and thus an irreducible representation r^0 of $\check{G}^0$. $\mathfrak{G}(\overline{\mathbf{Q}}/E)$ is the stabilizer of ρ^0 in $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$. Extend r^0 to $\check{G}^0 \times \mathfrak{G}(\overline{\mathbf{Q}}/E)$ by letting the elements of **[2]** $\mathfrak{G}(\overline{\mathbf{Q}}/E)$ act trivially on the highest weight vector. Then extend to a representation r of $\check{G}$ by induction.

Let M be the centralizer of $h(S)$. M is the Levi factor of a parabolic. Let T be a CSG of M and hence of G. Ω_G and Ω_M are the Weyl groups of T in G and M respectively. $\Omega_{\check{G}^0}$ and $\Omega_{\check{M}^0}$ also can be easily defined.

Lemma (proof omitted)**.**

(*i*) *The degree of* r *is*

$$[E:\mathbf{Q}] \deg r^0 = [E:\mathbf{Q}][\Omega_G : \Omega_M].$$

(ii) The stabilizer of the dominant weight of r^0 in $\Omega_{\check{G}^0}$ is $\Omega_{\check{M}^0}$ and $\Omega_{\check{G}^0}$ acts transitively on the weights of r^0. Observe that the E introduced above is the field over which the Shimura variety S_K is conjectured to be defined. Here

$$S_K(\mathbf{C}) = G(\mathbf{Q})\backslash G(\mathbf{A})/M(\mathbf{R})K \qquad K \subseteq G(\mathbf{A}_f).$$

Let s be a representation of G over $\mathbf{Q}$. I assume s is absolutely irreducible. As a consequence, s is trivial on the maximal anoisotropic torus in Z, the centre of G. As usual if the Shimura conjecture is valid s defines a sheaf $\mathcal{F}_s$ over S_K in the étale topology. I am in this letter always going to assume that S_K is proper over E. We are interested in the Hasse-Weil zeta-function $L(z, S_K, \mathcal{F}_s)$ as a formal, not as an analytic, object.

Let χ_∞ being an infinitely differentiable function in $G(\mathbf{R})$ with support which is compact modulo $Z^0(\mathbf{R})$ so that

(i)
$$\chi_\infty(zg) = s(z)\chi_\infty(g) \qquad z \in Z^0(\mathbf{R}).$$
$s(z)$ is here to be regarded as a scalar.

(ii) If A is a CSG of G over $\mathbf{R}$ not conjugate over $\mathbf{R}$ to a CSG of M and if $\gamma \in A$ is regular then **[3]**
$$\int_{A(\mathbf{R})\backslash G(\mathbf{R})} \chi_\infty(g^{-1}\gamma g)\, dg = 0.$$

(iii) If A is a CSG of M over $\mathbf{R}$ and if $\gamma \in A$ is regular in G then
$$\int_{A(\mathbf{R})\backslash G(\mathbf{R})} \chi_\infty(g^{-1}\gamma g)\, dg = \frac{\left|\Omega_G(\mathbf{R})\right|}{\left|\Omega_G(\mathbf{C})\right|}\frac{\operatorname{trace} s(\gamma)}{\operatorname{meas}\bigl(Z^0(\mathbf{R})\backslash A(\mathbf{R})\bigr)}.$$
Here $\Omega_G(\mathbf{C})$ is the Weyl group of T in $G(\mathbf{C})$ and $\Omega_G(\mathbf{R})$ the Weyl group in $T(\mathbf{R})$.

If π_∞ is an irreducible representation of $G(\mathbf{R})$ with
$$\pi_\infty(z) = s(z)^{-1} \qquad z \in Z^0(\mathbf{R})$$
set
$$\pi_\infty(\chi_\infty) = \int_{Z^0(\mathbf{R})\backslash G(\mathbf{R})} \chi_\infty(g)\pi_\infty(g)\, dg$$
and set
$$m(\pi_\infty) = \operatorname{trace}(\pi_\infty)(\chi_\infty).$$

Observe that no one has yet written out a proof of the existence of χ_∞. That it exists is not completely obvious. However, its existence in general should follow from some work that Harish-Chandra is now carrying out. For the groups in which I will eventually be concerned in this letter there is no problem.

Extend $s(z)$, $x \in Z^0(\mathbf{R})$ to a homomorphism $\overline{s} : G(\mathbf{A}) \to \mathbf{R}^+$ trivial on $G(\mathbf{Q})$. We are interested in the space of measurable functions φ on $G(\mathbf{Q})\backslash G(\mathbf{A})$ satisfying

(i)
$$\varphi(zg) = s(z)^{-1}\varphi(g) \qquad z \in Z^0(\mathbf{R})$$
and
$$\varphi(zg) = \varphi(g) \qquad z \in Z(\mathbf{A}_f) \cap K$$

[4]

(ii)
$$\int_{Z^0(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} \overline{s}(z)^2\left|\varphi(g)\right|^2 dg < \infty$$

if $Z^0(\mathbf{A}) = Z^0(\mathbf{R})\big(Z(\mathbf{A}_f) \cap K\big)$. The action R of $G(\mathbf{A})$ on this space is the direct sum of irreducible representations.

$$R = \bigoplus \pi.$$

All products below are taking over the constituents of R with multiplicity taken into account.

If $\pi = \pi_\infty \otimes \pi_f$ let $m(\pi_f, K)$ be the multiplicity with which the trivial representation of K occurs in π_f. Set

$$m(\pi, s, K) = m(\pi_\infty) m(\pi_f, K).$$

Let

$$q = \dim S_K.$$

The groups we shall be considering satisfy the following condition:

If $F \supseteq \mathbf{Q}$ is a field and if γ, γ' are regular in $G(F)$ and conjugate in $G(\overline{F})$ then they are conjugate in $G(F)$.

This assumption is very restrictive and guarantees that phenomena connected with L-indistinguishability do not occur. If it is satisfied $m(\pi, s, K)$ should always lie in $\mathbf{Z}$ and we should have

$$(*) \qquad \boxed{L(z, S_K, \mathcal{F}_s) = \prod_\pi L\Big(z - \frac{q}{2}, \pi, r\Big)^{m(\pi,s,K)}}$$

Before I begin the proof of this for the groups mentioned, let me hint at the type of modification it will require in general. By the way, I apologize for talking all the time about L-indistinguishability and yet giving you so little in the way of concrete results. Diana Shelstad and I have been trying to [**5**] analyze the phenomenon over $\mathbf{R}$ in what we hope will be a definitive way, especially in regard to applications to the trace formula. We are still having difficulty formulating correct general statements, let alone proving them, but I believe we are on the right track.

Suppose T is a torus in G over $\mathbf{Q}$ and κ is a character of $H^{-1}\Big(\check{L}(T)\Big)$. Here we use the fact that $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on

$$\check{L}(T) = \mathrm{Hom}\big(\mathrm{GL}(1), T\big).$$

Recalling the definition of associate group yields an isomorphism (not unique)

$$\check{L}(T) \simeq L(\check{T}).$$

Use it to identify the two groups. Let $\check{H}^0 \subseteq \check{G}^0$ be the subgroup of $\check{G}^0$ generated by $\check{T}$ and the one-parameter subgroups corresponding to those $\check{\alpha}$ with $\kappa(\check{\alpha}) = 1$. Write the action of σ on $\check{L}(T)$, and hence, because of the above identification, on $\check{T}$ as $\omega_1(\sigma)\omega_2(\sigma)$ where $\omega_2(\sigma)$ is inner with respect to $\check{H}^0$ and $\omega_1(\sigma)$ is outer, that is, leaves the *set* of roots of $\check{T}$ in $\check{H}^0$ positive with respect to some given order fixed. $\sigma \to \omega_1(\sigma)$ yields an action of $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ in the Dynkin diagram of $\check{H}^0$ and we can construct an associate group $\check{H}$. Assume for simplicity that $\check{H}^0 \hookrightarrow \check{G}^0$ extends to $\check{H} \hookrightarrow \check{G}$. This is by no means always so even if one replaces Galois groups by Weil groups. If it is not so, then complications arise which I am only beginning to understand.

Anyhow, as you know, $\check{H} \hookrightarrow \check{G}$ is it supposed to lead to maps $\pi' \to \pi$ from automorphic forms on H, the quasi-split group with $\check{H}$ as associate group, to automorphic forms on G. This is a rather complicated phenomenon. Fortunately as far as Shimura varieties are concerned it will be enough to have certain local information; the needed global facts will follow. The local information is still missing in general. Also r, a representation of $\check{G}$, pulls back to a representation r' of $\check{H}$. Although r is usually irreducible, **[6]** r' is usually reducible, and one can always pick out a certain component r'_0 of r'. Now

$$L(z,\pi,r) = L(z,\pi',r')$$

and $L(z,\pi',r'_0)$ is a factor of $L(z,\pi',r')$. In the case of a general Shimura variety the formula $(*)$ will probably have to be modified to include factors $L(z,\pi',r'_0)$. All this lies in the future.

At the moment I can, as I told you, only verify $(*)$ for almost all places, namely those for which

$$K = K^p K_p \qquad K^p \subseteq G(\mathbf{A}_f^p),\ K_p \subseteq G(\mathbf{Q}_p)$$

where K_p is a special maximal compact. To do this and take logarithms of both sides and look at the summand corresponding to the local factor at p.

Consider the right side first. Only those π occur for which π_p contains the trivial representation. For such π

$$m(\pi_f, K) = m(\pi_f^p, K^p)$$

and

$$L\Big(z-\frac{q}{2},\pi_p,r\Big) = \frac{1}{\det\Big(1-\frac{r\left(g(\pi_p)\right)}{p^{z-q/2}}\Big)}$$

if $g(\pi_p)$ is the conjugacy class of $\check{G}$ defined by π_p as in my Washington lecture.

$$\log L\Big(z-\frac{q}{2},\pi_p,r\Big) = \sum_{n=1}^{\infty}\frac{1}{np^{nz}}p^{nq/2}\operatorname{trace} r\big(g(\pi_p)\big)^n.$$

One of the main points of my Washington lectures was that there exists a homomorphism χ from the representation ring of $\check{G}$ (with a suitably restricted class of representations) to the Hecke algebra of $G(\mathbf{Q}_p)$ with respect to K_p. Then **[7]**

$$p^{nq/2}\operatorname{trace} r^n\big(g(\pi_p)\big)^n = \operatorname{trace}\pi_p\Big(\chi(p^{nq/2}r^{[n]})\Big).$$

Here $r^{[n]}$ has I hope the obvious meaning. It is an appropriate polynomial in the symmetric powers of r, the polynomial being defined by Newton's formulae. Note that $r^{[n]}$ is additive in r.

The local factor on the left side will be a product over the primes of E dividing p. Its logarithm will then be a sum. The corresponding decomposition of the right-hand side is furnished by a decomposition of r, now regarded as a representation of the local associate group. For the local associate group one replaces $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ by $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. The induced representation decomposes into a direct sum corresponding to the double coset decomposition

$$\mathfrak{G}(\overline{\mathbf{Q}}/E)\backslash\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})/\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p).$$

Each double coset also defines a prime of E dividing p. Let r_p be the restriction of r to the local associate group

$$r_p = \bigoplus r_p(i).$$

I want to show that

$$(**)\qquad \sum_{\pi} m(\pi, s, K) \log L\Big(z - \frac{q}{2}, \pi_p, r_p(i)\Big) = \log L_{\mathfrak{p}_i}(z, S_k, \mathcal{F}_s)$$

if $\mathfrak{p}_i$ is the prime corresponding to the ith double coset. With no loss of generality I may assume I am dealing with the coset containing the identity. $\mathfrak{p}$ will be the corresponding prime.

If $g = g_\infty g^p g_p$; $g^p \in G(\mathbf{A}_f^p)$, $g_p \in G(\mathbf{Q}_p)$, χ_p is the characteristic function of K^p divided by its measure, and $\chi_p^{(n)} = \chi\Big(p^{nq/2} r_p(i)^{[n]}\Big)$ set

$$\chi^{(n)}(g) = \chi_\infty(g_\infty)\chi^p(g^p)\chi_p^{(n)}(g_p).$$

[**8**] The coefficient of $1/np^{nz}$ on the left of $(**)$ is

$$\operatorname{trace} R(\chi^{(n)}).$$

This one computes by means of the Selberg trace formula. Proceeding as usual (cf. Antwerp), we obtain

$$\sum \operatorname{trace} \mu(\gamma) \frac{\operatorname{meas} Z^0(\mathbf{A})\backslash G_\gamma(\mathbf{A})}{\operatorname{meas} Z^0(\mathbf{A})\backslash G'_\gamma(\mathbf{A})} \cdot \left\{\int_{G_\gamma(\mathbf{A}_f^p)\backslash G(\mathbf{A}_f^p)} \chi^p(g\gamma g^{-1})\, dg\right\}\left\{\epsilon(\gamma)\int_{G_\gamma(\mathbf{Q}_p)\backslash G(\mathbf{Q}_p)} \chi_p^{(n)}(g\gamma g^{-1})\, dg\right\}.$$

The sum is over conjugacy classes in $G(\mathbf{Q}) \cap Z^0(\mathbf{A})\backslash G(\mathbf{Q})$ which are conjugate in $G(\mathbf{R})$ to an element of $M(\mathbf{R})$. G'_γ is the twisted form of G_γ over $\mathbf{R}$ with anisotropic derived group. $\epsilon(\gamma)$ is 1 if γ is not central. Otherwise it is $(-1)^d$ if d is the number of real places at which the quaternion algebra splits. G_γ is of course the centralizer of γ.

It is clear that we have to be able to handle the orbital integrals

$$\int_{G_\gamma(\mathbf{Q}_p)\backslash G(\mathbf{Q}_p)} \chi_p^{(n)}(g\gamma g^{-1})\, dg.$$

As I shall point out in the moment this is easy if γ is regular and lies in a CSG with maximal split part. Otherwise it is a difficult problem. Of course in the present case $G(\mathbf{Q}_p)$ is, since we are only working at good p, a product of GL(2)'s over unramified extensions of $\mathbf{Q}_p$; so that all the necessary computations have been carried out my Antwerp report. These I put together in the form needed for the present purposes in my letter of October, 1973 to which I refer you. If you find that letter too laconic let me know and I will amplify.

That the integrals are easily computed when γ is regular and lies in a CSG with maximal split part is a consequence of the definitions of my Washington lecture. If T is such a CSG there is a homomorphism

$$\check{\lambda} : T(F) \to L(\check{T})$$

[**9**] with

$$\big|\lambda(\gamma)\big| = p^{-\langle \lambda, \check{\lambda}(\gamma)\rangle} \qquad \lambda \in L(T)$$

$\check{\lambda}(\gamma)$ is always invariant under $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$. If σ is the Froebenius, $t \in \check{T}(\mathbf{C})$, and X is an element of the representation ring of $\check{G}$ then

$$\operatorname{trace} X(t \times \sigma) = \sum a(\check{\lambda})\check{\lambda}(t) \qquad t \times \sigma \in \check{G}$$

where the sum runs over the invariant elements in $\check{L}(T)$. The coefficients $a(\check{\lambda})$ depend on X. If $f = \chi(X)$ then

$$\prod_{\alpha>0} \left|1-\alpha(x)\right|\left|\alpha(x)\right|^{-1/2} \int_{T(\mathbf{Q}_p)\backslash G(\mathbf{Q}_p)} f(g\gamma g^{-1})\, dg$$

is equal to

$$c(G)a\Big(\check{\lambda}(\gamma)\Big).$$

$c(G)$ is a constant which depends on G alone.

Let κ be the residue field of E at $\mathfrak{p}$. I shall first explain the conjectures about the structure of $S_K(\overline{\kappa})$ mean in the present circumstances, then I shall review what I said in my old letter about the combinatorial facts to be proved, then I shall carry out the necessary combinatorial analysis. I first of all correct (once again!) the definition of equivalence of two pairs (γ_1, h_1^0), (γ_2, h_2^0) of Frobenius type. The relations

$$\gamma_2^m = g\gamma_1^m g^{-1}$$

on p. 20 and on p. 21 of my Rapoport letter should only hold modulo the centre

$$\gamma_2^m = zg\gamma_1^m g^{-1} \qquad z \in Z(\mathbf{Q}).$$

Moreover I should not pass to the direct limit. The spaces [**10**]

$$H(\mathbf{Q})\backslash G(\mathbf{A}_f^p) \times X/K^p$$

make sense but

$$H(\mathbf{Q})\backslash G(\mathbf{A}_f^p) \times X$$

does not. I am consistently confused about the anisotropic part of the centre.

Suppose (γ, h) is of Frobenius type. Then γ is elliptic at infinity. There are two possibilities:

(i) No power of γ lies in F, the totally real field over which the quaternion algebra is given. In this case $F(\gamma)$ is a totally imaginary quadratic extension F' of $\mathbf{Q}$.

(ii) Some power of γ lies in F.

Recall that we fix

$$\begin{array}{ccccc} E & \subseteq & \overline{\mathbf{Q}} & \subseteq & \mathbf{C} \\ & & & \rotatebox{45}{$\subseteq$} & \\ & & & & \overline{\mathbf{Q}}_p \end{array}.$$

The imbedding $E \hookrightarrow \overline{\mathbf{Q}}_p$ is to define $\mathfrak{p}$. The imbeddings of F in $\overline{\mathbf{Q}}$ or $\mathbf{C}$ are parameterized by the homogeneous space

$$\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})\backslash\mathfrak{G}(\overline{\mathbf{Q}}/F).$$

We write this set as

$$\times\times\cdots\times \quad \circ\cdots\circ\times\cdots.$$

A cross represents a place at which the algebra splits; a circle a place at which it does not. We break it up into orbits under the Frobenius

$$\underbrace{\times\cdots\circ\cdots\times}_{n_1};\underbrace{\times\cdots\circ\cdots\times}_{n_2};\cdots.$$

Let n_i be the number of elements in the ith orbit. For concreteness, [**11**] suppose the Frobenius operates cyclically on each of the orbits. Let $n = \sum n_i$. The cocharacter $\check{\mu}$ determined by h^0

has n coordinates which we take, with an obvious notation, to be integral diagonal matrices. At an unmarked point ($\circ$)

$$\check{\mu}_j = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

At a marked point ($\times$)

$$\check{\mu}_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$\check{\nu} = \mathrm{Nm}\,\check{\mu}$$

where the norm is taken say with respect to a large extension k_p of degree m. Let b_i be the number of marked points in the ith orbit. Then $\check{\nu}_j$ depends only on the orbit to which j belongs.

$$\check{\nu}_j = \begin{pmatrix} k_i & 0 \\ 0 & k_i' \end{pmatrix}$$

with

$$k_i + k_i' = \frac{mb_i}{n_i}.$$

If F' is not split at the prime of F corresponding to the ith orbit, then $k_i = k_i'$. If some power of γ lies in F then $k_i = k_i'$ for all i. It follows easily that if we are truly dealing with case (i) then the field F' is split at at least one prime dividing p. Moreover, $k_i \neq k_i'$ for at least one i.

The condition on equivalence of (γ_1, h_1^0), (γ_2, h_2^0) away from p just **[12]** means, for the groups under consideration, that γ_1 and γ_2 are conjugate modulo the centre. To see what the condition at p means we find the F defined on p. 24 of the Rapoport letter. In the meantime I have begun to denote this F by the new name b. Let me do so here also. The construction of b is such that it respects any decomposition of G into a product over $\mathbf{Q}_p$. Thus with no loss of generality I may assume there is only one orbit.

(a) Suppose we are in case (i) and F' splits over F. We may take k_p unramified. Then b is of the form

$$b = \big(b(1), \dots, b(n)\big) \qquad b(i) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

We are interested in b modulo the relation $b \sim cb\sigma(c^{-1})$. This means that the only invariants are the absolute values of the diagonal elements of

$$\prod_i b(i).$$

Take $w = (1, \sigma)$ in the Weil group as on p. 14 of the Appendix and use b_w to define b. Then $b_w = a_\sigma$ and

$$b(1) = \begin{pmatrix} p^{\alpha_{i+1}} & 0 \\ 0 & p^{\beta_{i+1}} \end{pmatrix} \qquad \check{\mu}_{i+1} = \begin{pmatrix} \alpha_{i+1} & 0 \\ 0 & \beta_{i+1} \end{pmatrix}.$$

Thus

$$\prod b(i) = \begin{pmatrix} p^k & 0 \\ 0 & p^{k'} \end{pmatrix}.$$

Here $k = k_1$ (there is now only one orbit) is the number of $\check{\mu}_j$ of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

(b) Suppose we are in case (i) and F' does not split over F or we are in case (ii). In the second case we have to pick an anisotropic CSG to define **[13]** b. This is just the same as picking a field F' which does not split at p. So we can treat both cases uniformly, let $\mathfrak{k}$ be the completion of the maximal unramified extension of $\mathbf{Q}_p$ and let $\mathfrak{k}'$ be the composite of F' and $\mathfrak{k}$. Let $n' = n$ if F' is ramified and $n' = 2n$ if E is unramified at p. We write the elements of $(F' \otimes_{\mathbf{Q}_p} \mathfrak{k})^k$ as

$$d = \big(d(1), \dots, d(n')\big) \qquad d(i) \in \mathfrak{k}'.$$

The action of the Frobenius on the second factor of $F' \otimes \mathfrak{k}$ is given by

$$\big(d(1), \dots, d(n')\big) \to \Big(\sigma^{n'} d(n'), d(1), \dots, d(n-1)\Big).$$

$\sigma^{n'}$ is understood to be extended to $\mathfrak{k}'$ so that it acts trivially on F'. Thus if

$$b = \big(b(1), \dots, b(n')\big)$$

all that matters is

$$b' = \prod b(i)$$

modulo the equivalence $b' \sim cb\sigma^{n'}(c^{-1})$. Thus only $|b'|$ matters or rather $\big|\lambda(b)\big|$ if λ is a character of the CSG corresponding to F' defined over $\mathbf{Q}_p$. Any such character is a multiple of

$$\lambda_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

As on p. 24 of the Rapoport letter

$$\big|\lambda(b)\big| = p^{-\langle \lambda, \check{\mu} \rangle}.$$

In particular **[14]**

$$\big|\lambda_0(b)\big| = p^{-\langle \lambda_0, \check{\mu} \rangle} = p^{-b}$$

if b is the number of marked places.

We now return to the original situation, that is, p is once again not assumed to remain prime in F. The equivalence classes of Frobenius pairs (γ, h^0) can now be described.

(i) Let F' be a totally imaginary quadratic extension of F which splits over at least one prime of F dividing p. Let S be a nonempty subset of the primes dividing p at which E splits so that b_i is even outside of S. For each $\mathfrak{p}_i \in S$ let $k_i \geqslant 0$, $k'_i \geqslant 0$, with $k_i \neq k'_i$ be integers so that $k_i + k'_i = b_i$, the number of marked places in the ith orbit. To this data corresponds an equivalence class of type (i). All equivalence classes of type (i) are obtained in this way. Because of the automorphism of F' over F the classes corresponding to $\Big(F', \big\{(k_i, k'_i)\big\}\Big)$ and $\Big(F', \big\{(k'_i, k_i)\big\}\Big)$ are the same.

(ii) There is one class of type (ii).

Notice that this corresponds exactly to the grouping of terms in my old letter. However, in the letter I took $k_i \leqslant b_i/2$ so that I had to introduce the multiplicity 2^{s-1} if $|S| = s$.

Let me come to the construction of X. The F_0 referred to on p. 25 of the Rapoport letter is given by

$$g \to b\sigma(g)$$

if $g \in G(\mathfrak{k})$ represents a point of X''. Observe that in the definition of X, y' should be $F_0 x'$ not $F_0^r x'$. You might find the enclosed lecture useful; it does not contain so many annoying slips. In the lecture F_0 is denoted F. Let me use this notation here also except that I make F boldface, thus $\mathbf{F}$. The construction of X respects products so I may as well work again at one fixed prime dividing p. Denote the X'' of the Rapoport letter by $\mathcal{H}$ as in the lecture. If σ is the ring of integers in $\mathfrak{k}$, the maximal unramified extension of $\mathbf{Q}_p$ then a point of $\mathcal{H}$ is just an n-tuple [**15**]

$$(M_1, \dots, M_n)$$

of two-dimensional σ-lattices. We may so arrange matters that the action of the Frobenius σ is

$$\sigma : (M_1, \dots, M_n) \to ({}^\sigma M_n, {}^\sigma M_1, \dots, {}^\sigma M_{n-1}) \qquad (n = n_i).$$

${}^\sigma M_j$ is obtained from M_j by letting σ act on the coordinates of its elements. We may suppose

$$b = (d, 1, \dots, 1)$$

so that

$$\mathbf{F} : (M_1, \dots, M_n) \to (d\ {}^\sigma M_n, {}^\sigma M_1, \dots, {}^\sigma M_{n-1}).$$

If we are working with an equivalence class of the first type and $\mathfrak{p}_i \in S$ we may diagonalize F' and take

$$d = \begin{pmatrix} p^k & 0 \\ 0 & p^{k'} \end{pmatrix} \qquad k \neq k'.$$

In this case $\overline{G}_i = \overline{G}_i^0$ is the torus corresponding to F' in G_i (the ith factor of G over FQ_p). Otherwise $\overline{G}_i^0 = G_i$. If F' splits over $\mathfrak{p}_i$ then

$$d = \begin{pmatrix} p^{b_i/2} & 0 \\ 0 & p^{b_i/2} \end{pmatrix}$$

Consequently $\overline{G}_i$ is also G_i. If we are dealing with a prime $\mathfrak{p}_i$ at which E does not split or we are dealing with case (ii) only the order of $\det d$ matters. This order is b_i. Thus $\overline{G}_i$ is G_i or the multiplicative group of a quaternion algebra over $F_{\mathfrak{p}_i}$ according as b_i is even or odd. This all fits in perfectly with my old letter. In these cases in which only the order of d matters I take [**16**]

$$d = p^n I \text{ or } d = p^n \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

This is mostly just to be definite.

Let $\overline{M}_i$ be the image of M_i in the Bruhat-Tits building of SL(2). Thus $\overline{M}_i$ is M_i up to similarity. Define M_i, $i \in \mathbf{Z}$ by the periodicity condition

$$M_{i+n} = d^{-1} M_i$$

$M \in \mathcal{H}$ defines a point of X if and only if $\overline{M}_i = \sigma(M_{i-1})$ when i is an unmarked point ($\circ$) and $\overline{M}_i$, $\sigma(M_{i-1})$ are joined by an edge when i is a marked point ($\times$).

Before analyzing X more carefully let's recall what we need to know. Just as in the Antwerp report the Lefschetz formula comes down to looking at the action of powers $\mathbf{F}^m$ of $\mathbf{F}$ on sheaves over the discrete spaces

$$Y_K = H(\mathbf{Q}) \backslash G(\mathbf{A}_f^p) \times X / K^p.$$

A lemma similar to those proved in §5 of the Antwerp report, the result of which is summarized on the fourth line of p. 9 of my old letter, shows that it is all a matter of computing certain class functions $\varphi^m(\gamma)$ on $\overline{G}(\mathbf{Q}_p)$. By the way, the sum on p. 9 just mentioned is over conjugacy classes in H. If you have any trouble deriving this formula, I would be glad to give further details. I also remark that $m > 0$ is to be such that the Frobenius element σ^m lies in $\mathfrak{G}(\overline{\mathbf{Q}}/E)$. Thus a cyclic shift by m takes a marked place to a marked place.

Since I have changed the action of $\overline{G}(\mathbf{Q}_p)$ in X from right to left let me redefine $\varphi^m(\gamma)$. If $x \in X$, set

$$T_x^m = \left\{ g \in \overline{G}(\mathbf{Q}_p) \;\middle|\; \mathbf{F}^m x = gx \right\}$$

and let δ_x^m be the characteristic function of T_x^m. If $\{x_i\}$ is a set of representatives for the orbits of $\overline{G}(\mathbf{Q}_p)$ in X set [**17**]

$$\varphi^m(\gamma) = \sum_i \frac{1}{\operatorname{meas} \overline{G}_{x_i}(\mathbf{Q}_p)} \int_{\overline{G}_\gamma(\mathbf{Q}_p)\backslash \overline{G}(\mathbf{Q}_p)} \delta_{x_i}^m(h^{-1}\gamma h)\, dh.$$

It will follow from what we do below that integrals are finite and that for a given m all but finitely many $\delta_{x_i}^m$ are identically zero. It is, however, not in general true, contrary to an assertion of the letter, that there are only finitely many orbits.

As explained rather briefly in the letter all we have to do is show that the functions $\varphi^m(\gamma)$ can be given explicitly by certain formulae. The factorization of G and of $\overline{G}$ over $\mathbf{Q}_p$ leads to a factorization

$$\varphi^m(\gamma) = \prod_i \varphi_i^m(\gamma_i).$$

It is enough to describe the formulae for the φ_i^m. I repeat them below for convenience.

I find my notes a little difficult to decipher so I will not try to develop a method which will apply uniformly, but rather use the same decomposition into cases as in the letter. This will make the exposition longer, but perhaps more digestible.

There are some general remarks to be made. A point in X is given by a sequence of lattices, M_j, $j \in \mathbf{Z}$, satisfying

(i) $\sigma(\overline{M}_{j-1}) = \overline{M}_j$ if i is unmarked.
(ii) $\sigma(\overline{M}_{j-1})$ and $\overline{M}_j$ are joined by an edge if i is a marked point.
(iii) $dM_{j+n_i} = M_j$.

After writing down these properties I realize they are insufficient. What is missing is the condition coming from the abelian part which I said on p. 25 of the Rapoport letter I could not remember. For a torus the reciprocity map at a place $\mathfrak{p}$ of E with a uniformizing parameter $\varpi(= p)$ is given by $\varpi \to t$ with [**18**]

$$\lambda(t) = \prod_{\mathfrak{G}(\overline{\mathbf{Q}}_p/E_\mathfrak{p})\backslash \mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)} \varpi^{\langle \tau\lambda, \check{\mu}\rangle}$$

if we are dealing with a prime at which everything is unramified. If

$$r = [E_\mathfrak{p} : \mathbf{Q}_p]$$

then

$$t = \prod_{j=0}^{r-1} \varpi^{\sigma^j \check{\mu}}.$$

On the other hand if $w_0 = (1, \sigma)$ belongs to the Weil group then

$$b_{w_0} = \sum_{j=0}^{r-1} \sigma\sigma^j \check{\mu} \otimes a_{\sigma,\sigma_j} = \check{\mu} \otimes \varpi = \varpi^{\check{\mu}}$$

and

$$b_{w_0}r = \prod_{j=0}^{r-1} \varpi^{\sigma^j \check{\mu}}.$$

This will of course establish the validity of my conjecture for tori. It also shows what condition is missing.

If x is the point of $\mathcal{H}$ represented by g and λ is a rational character of G over $\mathfrak{k}$ then

$$\bigl|\lambda(g)\bigr| = |\lambda|(x)$$

depends only on x. The extra condition to be imposed for x to lie in X is that

$$\frac{|\lambda|(F_x)}{|\lambda|(x)} = |p^{\langle\lambda,\check{\mu}\rangle}|$$

for all such λ. The reason that the omission of this condition caused no embarrassment in the Rapoport letter is that it actually appeared surreptitiously in the assertion on p. 53 that the difference between the orders of $e\widehat{M}$ and $e\mathbf{F}\widehat{M}$ is $a(e)$.

[**19**] In the present case it allows (i) and (ii) to be strengthened to:

(i′) $\sigma(M_{j-1}) = M_j$ if j is unmarked.
(ii′) $M_j \supsetneqq \sigma(M_{j-1}) \supsetneqq pM_j$ otherwise.

It will be simpler if we dispose immediately of the trivial case that there are no marked points in the orbit. Then $\overline{G}_i(\mathbf{Q}_p) = \mathrm{GL}(2, F_{\mathfrak{p}_i})$ and all the M_i are determined by M_0 which must be a lattice over $F_{\mathfrak{p}_i}$. There is only one orbit. A representative x_0 is obtained by taking M_0 as the lattice V of all integral vectors. Then $T^m_{x_0} = \mathrm{GL}(2, \mathcal{O}_{\mathfrak{p}_i})$ and

$$\varphi^m_i(\gamma) = \frac{1}{\operatorname{meas}\mathrm{GL}(2,\mathcal{O}_{\mathfrak{p}_i})} \int_{\mathrm{GL}(2,F_{\mathfrak{p}_i})_\gamma\backslash \mathrm{GL}(2,F_{\mathfrak{p}_i})} \delta^m_{x_0}(h^{-1}\gamma h)\, dh.$$

This is the function demanded by the trace formula. In the letter I gave explicit formulae for the integral, but they are not always necessary.

We begin with classes of type (i) taking $k_i \neq k'_i$. Let ℓ_i be the greatest common divisor of m and n_i and let

$$\frac{b_i}{a_i} = \frac{n_i}{\ell_i}.$$

Notice that if we take the orbits of σ^m in the ith orbit under σ then such an orbit consists entirely of marked or entirely of unmarked points. The number of orbits consisting of marked points is a_i. If $n_i/\ell_i|k_i$, let η be the characteristic function of those

$$\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

in $\overline{G}_i(\mathbf{Q}_p)$ for which

$$|\alpha| = |p^{mk_i/n_i}| \qquad |\beta| = |p^{mk'_i/n_i}|.$$

Then [**20**]

$$\operatorname{meas}\overline{G}_i(\mathbf{Z}_p)\varphi^m_i(\gamma) = \begin{cases} 0 & n_i/\ell_i \nmid k_i \\ a_i & d_i = \min\{k_i, k'_i\} \\ \left(\frac{a_i k_i}{b_i}\right)^{p^{md_i}\eta(\gamma)} & n_i/\ell_i | k_i \end{cases}$$

is the form predicted for φ^m_i on p. 9 of the old letter. $\overline{G}_i(\mathbf{Z}_p)$ has I hope an obvious meaning.

Let

$$x = \{M_j\}.$$

and set

$$L_j = \sigma^{-j}(M_j).$$

Then

$$dL_{j+n_i} = \sigma^{-n_i}(L_j).$$

If

$$\mathbf{F}^m x = x' = \{M'_j\}$$

then

$$M'_j = \sigma^m(M_{j-m})$$

and

$$L'_j = L_{j-m}.$$

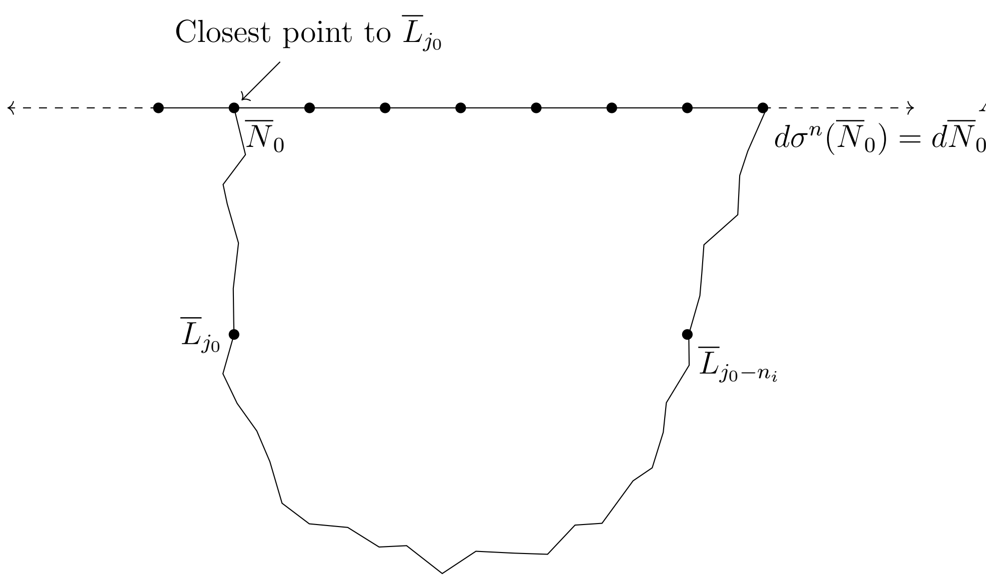
We suppose the torus T_i defined by F' in G_i is the group of diagonal matrices. If $\gamma \in T_x^m$ then **[21]**

$$L_{j-m} = \gamma L_j.$$

Let $\overline{L}_j$ be the image of L_j in the Bruhat-Tits building of $\mathrm{SL}(2, \mathfrak{k})$. Since $\overline{L}_j = \overline{L}_{j-1}$ or $\overline{L}_j$ and $\overline{L}_{j-1}$ are joined by an edge the sequence $\{\overline{L}_j\}$ defines a path, perhaps infinite, in this Bruhat-Tits building. Consider the apartment A of the building corresponding to the diagonal matrices. Amongst all the $\overline{L}_j$ there is one $\overline{L}_{j_0}$ whose distance from this apartment is minimal. Since $\sigma^{-j_0}(\gamma)$ is also a diagonal matrix, as is d,

$$\rho(\overline{L}_{j_0}, A) = \rho(\overline{L}_{j_0-m}, A) = \rho(L_{j_0+n_i}, A).$$

Picture



Because the Bruhat-Tits building is a tree we conclude that $\overline{L}_{j_0} \in A$. Observe that if $\overline{L}_j \in A$, so is $\overline{L}_{j-m}$ and $\overline{L}_{j+n_i}$, and hence so is $\overline{L}_{j+\ell_i}$. Since we are only interested in orbits under $\overline{G}_i(\overline{\mathbf{Q}}_p)$ we may always assume that for at least one j_0, L_{j_0} is the standard lattice V of integral vectors. Observe that

$$L_{j_0-mn_i} = d^m L_{j_0} = \gamma^{n_i} L_{j_0}$$

and

$$d^m \gamma^{-n_j}$$

stabilizes V. Thus **[22]**

$$|\alpha| = |p^{mk_i/n_i}| \qquad |\beta| = |p^{mk_i'/n_i}|.$$

Also mk_i/n_i must be integral so $n_i/\ell_i | k_i$.

The skeleton will be the set of all j such that $\overline{L}_j \in A$. Observe that for all j

$$d^{m/\ell_i}\sigma^{mn_i/\ell_i}(L_j) = L_{j-mn_i/\ell_i} = \gamma^{n_i/\ell_i} L_j$$

or

(1) $$\sigma^{mn_i/\ell_i}(L_j) = d^{-m/\ell_i}\gamma^{n_i/\ell_i} L_j.$$

If $um + vn_i = \ell_i$ then

$$L_{j-\ell_i} = L_{j-um-vn_i} = \gamma^u L_{j-vn_i} = \gamma^u d^v \sigma^{vn_i}(L_j)$$

or

(2) $$L_{j-\ell_i} = \gamma^u d^v \sigma^{vn_i}(L_j).$$

From (1) and (2) we deduce conversely

$$\begin{aligned} L_{j-n_i} = L_{j-\frac{n_i}{\ell_i}\ell_i} &= \gamma^{un_i/\ell_i} d^{vn_i/\ell_i} \sigma^{vn_i^2/\ell_i}(L_j) \\ &= d\gamma^{un_i/\ell_i} d^{-mu/\ell_i} \sigma^{n_i - \frac{um}{\ell_i} n_i}(L_j) \\ &= d\sigma^{n_i}(L_j) \end{aligned}$$

and **[23]**

$$\begin{aligned} L_{j-m} = L_{j-\frac{m}{\ell_i}\ell_i} &= \gamma^{um/\ell_i} d^{vm/\ell_i} \sigma^{vn_i m/\ell_i}(L_j) \\ &= \gamma\gamma^{-vn_i/\ell_i} d^{vm/\ell_i} \sigma^{vn_i m/\ell_i}(L_j) \\ &= \gamma L_j. \end{aligned}$$

If we manage to define the chain for $\ell_i + 1$ consecutive values of j so that (1) is satisfied, so that the periodicity condition (2) is satisfied for the two end points and so that the conditions

(i′) $L_{j-1} = L_j$ if j is unmarked.

(ii′) $L_j \supsetneqq L_{j-1} \supsetneqq pL_j$ if j is marked.

are satisfied for all j but the first, where they are inapplicable, then we can extend by periodicity to obtain a point of X. Actually it is enough to arrange the barred form of these two conditions for once L_{j_0} is fixed there will be a unique lifting.

Before proceeding further there are two general remarks to be made about the functions

$$\begin{aligned}\varphi^m(\gamma) &= \sum \frac{1}{\operatorname{meas} \overline{G}_x} \int_{\overline{G}_\gamma(\mathbf{Q}_p)\backslash \overline{G}(\mathbf{Q}_p)} \delta_x^m(h^{-1}\gamma h)\, dh \\ &= \sum \frac{1}{\operatorname{meas} \overline{G}_x} \sum_{\overline{G}_\gamma(\mathbf{Q}_p)\backslash \overline{G}(\mathbf{Q}_p)/\overline{G}_x} \delta_{hx}^m(\gamma) \operatorname{meas}\Big(\overline{G}_\gamma(\mathbf{Q}_p)\backslash \overline{G}_\gamma(\mathbf{Q}_p) h \overline{G}_x\Big)\end{aligned}$$

If $\overline{G}_{\gamma,x}$ is the stabilizer of x in $\overline{G}_\gamma$ then

$$\frac{\operatorname{meas}\Big(\overline{G}_\gamma(\mathbf{Q}_p)\backslash \overline{G}_\gamma(\mathbf{Q}_p) h \overline{G}_x\Big)}{\operatorname{meas} \overline{G}_x} = \frac{1}{\operatorname{meas} \overline{G}_{\gamma,hx}}.$$

[24] Thus $\varphi^m(\gamma)$ is the sum over the orbits of $\overline{G}_\gamma(\mathbf{Q}_p)$ of

$$\frac{\delta_x^m(\gamma)}{\operatorname{meas} \overline{G}_{\gamma,x}}.$$

Fix γ and consider the set U of all x for which $\delta_x^m(\gamma) = 1$. Suppose $U' \subseteq U$ and $\overline{G}_0$, an open subgroup of $\overline{G}_\gamma(\mathbf{Q}_p)$, are such that

(i) Every orbit in U meets U'.
(ii) If x and y in U' lie in the same orbit of $\overline{G}_\gamma(\mathbf{Q}_p)$ then $x = gy$, $g \in \overline{G}_0$.
(iii) For all $x \in U'$, $\overline{G}_x$ is a subgroup of $\overline{G}_0$.

Then

$$\sum \frac{\delta_x^m(\gamma)}{\operatorname{meas} \overline{G}_{\gamma,x}} = \frac{|U'|}{\operatorname{meas} \overline{G}_0}$$

if $|U'|$ is the cardinality of U'.

In the case we are treating at present U' can be taken as follows. In each subset X which appears as a rational skeleton we fix a $j_0 = j_0(X)$. U' is the set of all $x = \{M_i\} \sim \{L_i\}$ such that L_{j_0} is the standard lattice V if $j_0 = j_0(X)$, X the skeleton of x.

Observe that if

$$\gamma^u d^v = \begin{pmatrix} \alpha' & 0 \\ 0 & \beta' \end{pmatrix}$$

then

$$|\alpha'| = \left| p^{\frac{\ell_i k_i}{n_i}} \right| \qquad |\beta'| = \left| p^{\frac{\ell_i k_i'}{n_i}} \right|.$$

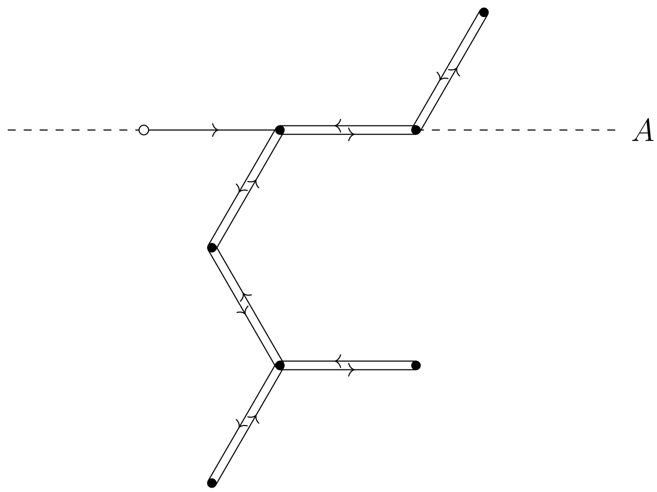
Thus the condition (2) demands that the path in A formed from the $\overline{L}_j$ with j in the rational skeleton joins the points $\overline{L}_{j_0}$, $\overline{L}_{j_0+\ell_i}$ whose distance apart is **[25]**

$$\left| \frac{\ell_i}{n_i}(k_i - k_i') \right| = e.$$

If f is the number of marked points in the skeleton between j_0 and $j_0 + \ell_i - 1$ inclusive the length of the path is f. Observe that this subpath is obtained from the original path simply by discarding all edges not in A. Let N_f^e be the number of paths in A of length f joining two points whose distance apart is e.

To simplify I may as well discard all unmarked places. They obviously play no role. However, I must then replace u_i by b_i and ℓ_i by a_i. I do this and now suppose all places are marked. Suppose X is the rational skeleton of x. X has f elements (in an interval of length a_i of course) and $a_i - f = 2c$ is even. I want to introduce a set with $c+f$ elements. Let r be the number of gaps that occur when the points not in the rational skeleton are removed. The length of each gap is even. Let the lengths be $2s_1, \ldots, 2s_r$.

Example



$$e = 1 \qquad f = 3$$
$$r = 2$$
$$s_1 = 4 \qquad s_2 = 1$$

I refer to the r subpaths as digits. Let T_0 be the tree formed by the non-negative real numbers with non-negative reals as vertices. To each digit I associate a path in T_0 beginning and ending at zero. The length of the path will be $2s_i$. I make the definition by example. Consider the first digit in the above example. The associated path goes directly out to three, then back to two, then out to three, then directly back to zero. Thus to each edge of the [**26**] digit corresponds an edge of the path in T_0. The edge of the digit will be called progressive or retrogressive according as the edge in T_0 in leading away or towards 0. To the points of the skeleton we add all points j such that $\overline{L}_j$, $\overline{L}_{j+1}$ are joined by a progressive edge. The new set has $c+f$ elements. Call it the frame Y.

Observe that $f = e + 2f'$ and that $c + f = c' + f' + e$ if $a_i = 2c' + e$. Supposed we are given a subset Y with $c' + f' + e$ elements $f' \geqslant 0$. (We always work modulo a_i.) Set

$$N_{j_1,j_2} = \sum_{j=j_1}^{j_2} \epsilon_j \qquad j_2 \geqslant j_1$$

where ϵ_j is 1 or -1 according as j lies or does not lie in Y. Set

$$X = \bigl\{ j_1 \bigm| N_{j_1,j_2} \geqslant 0 \text{ for } j_2 \geqslant j_1 \bigr\}.$$

Observe this process does in fact allow us to pass back from the frame to the skeleton. I claim that in general X, which I may call the skeleton of Y, is not empty. I use induction on c'. Y is not empty for $c' + e > 0$. Choose $j_1 \in Y$. If $j_1 \in X$ we are done. Otherwise there is a smallest j_2 such that $N_{j_1,j_2} < 0$. Then $j_2 > j_1 + 1$ for $N_{j_1,j_1} = 1$. Also $N_{j_1,j_2-1} = 0$ so $j_2 - j_1 - 1$ even. Discard $j_1, \ldots, j_2 - 1$. We obtain a set with $a_i - (j_2 - j_1) - 1$ elements a subset Y' with $c' - (j_2 - j_1 - 1) + e/2$ elements, because exactly half of the discarded elements lie in Y. Since $j_2 < j_1 + a_i$ we may apply a induction, for our assertion is clear if $c' = 0$ or $e = 0$, $c' = 1$. The skeleton of Y' is clearly contained in the skeleton of Y. The index j_1 will be called an extremity of the skeleton if it lies in the skeleton and $N_{j_1,j_2} = 0$ for some $j_2 \geqslant j_1$. The skeleton does have extremities unless $f' = c'$ for if j_1 lies in the skeleton and is not an extremity then $j_1 + 1$ also lies in the skeleton.

We shall also call the subset Y a frame. We associate to a frame: **[27]**

1. The skeleton.
2. The number r of extremities.
3. To each extremity j_α the number $2s_\alpha$ chosen to be minimal so that
$$N_{j_\alpha, j_\alpha+2s_\alpha-1} = 0.$$
4. The spine, which is obtained from the skeleton by discarding the extremities.

I claim that the spine contains $2f'+e$ points. This is clear if $f'=c'$; otherwise we proceed by induction. Let j_α be an extremity. Remove $j_\alpha, j_\alpha+1, \dots, j_\alpha+2s_\alpha-1$. If $2s_\alpha = 2c'+e$ the result is clear. Otherwise a_i is replaced by $a_i - 2s_\alpha$, and r is reduced by 1, for no new extremities are introduced and the spine is unchanged. It follows by induction that
$$a_i = 2f' + e + \sum 2s_\alpha$$
and that the spine contains $2f'+e$ points.

Suppose we have a frame and a given path joining $\overline{L}_{j_0}$ and $\overline{L}_{j_0+a_i}$ corresponding to its skeleton. We ask ourselves how many elements of U' correspond to this frame and this path. It is a matter of counting the number of possible digits corresponding to a given gap of length $2s_\alpha$. For the retrogressive edges there is no choice; in them we are just retracing our path.

Set
$$B = d^{-m/\ell_i}\gamma^{n_i/\ell_i}.$$
Suppose a given lattice L satisfies
$$BL = \sigma^{mn_i/\ell_i}(L).$$
[28] Then how many L' with $L \supsetneqq L' \supsetneqq pL$ are there with $BL' = \sigma^{mn_i/\ell_i}(L)$? Since $\sigma^{mn_i/\ell_i}(B) = B$ we can regard the map $\lambda \to B^{-1}\sigma^{mn_i/\ell_i}(\lambda)$ as defining a vector space structure on L/pL over the field with p^{mn_i/ℓ_i} elements. Since there is only one such structure it follows that there are $p^{mn_i/\ell_i}+1$ possibilities for L'. At a progressive edge we pass from such an L to such an L'. Since the only thing we cannot do at a progressive edge is turn back, this yields p^{mn_i/ℓ_i} possibilities. However, at the first of the progressive edges there are two forbidden directions, those which lie in A. Since $n_i/\ell_i = b_i/a_i$ we conclude that

$$|U'| = \sum \left(1 - \frac{1}{p^{\frac{mb_i}{a_i}}}\right)^r p^{\frac{mb_i}{a_i}s} N_f^e. \tag{3}$$

Here the sum is over all subsets of the integers modulo a_i with at least $c'+e$ elements. If the number of elements is $c'+f'+e$ then $f = 2f'+e$ and $s = \sum_{\alpha=1}^r s_\alpha$. What we have to do is show that
$$|U'| = \binom{a_i}{\frac{a_ik_i}{b_i}} p^{md_i} \qquad d_i = \min\{k_i, k_i'\}.$$
With no loss of generality we may suppose $k_i \leqslant k_i'$ so that $c' = a_ik_i/b_i$. For the sake of later applications we now allow the possibility that $k_i = k_i'$. This means $e=0$.

The expression (3) is equal to
$$\sum_{j=0}^{c'} \left(\sum_{r \geqslant j} \binom{r}{j} (-1)^j N^e_{a_i-2s} \right) p^{mb_i/a_i(s-j)}.$$

The inner sum is over all Y for which $r \geqslant j$. To prove that it has the correct value we perform a preliminary reduction. Suppose we have a frame with **[29]** $a_i - s$ elements and the corresponding $r \geqslant j$. Then $s \geqslant j$. We can construct $\binom{r}{j}$ subsets with $a_i - (s-j) = a_i - s + j$ elements as follows. Take a subset $\{1, \dots, r\}$ with j elements. For each α in this subset consider the gap $j_\alpha, \dots, j_\alpha + 2s_\alpha - 1$ and for each α add $j_\alpha + 2s_\alpha - 1$ to the subset. The new subset will have $a_i - s + j$ elements. Moreover the added points will not be extremities, for if j lies in the skeleton of the old it lies in the skeleton of the new and $j-1$ is certainly no extremity. Moreover for each α in the subset j_α is an extremity of the old frame but not of the new.

We ask ourselves the following question: How often is a given frame with $a_i - s + j$ elements obtained by the above procedure? Of course the procedure yields the frame together with j distinguished elements of the skeleton which are not extremities. The spine has $a_i - 2s + 2j$ elements. Also these j points of the spine are each separated by some other point of the spine. Suppose we start with such a situation. Remove the j distinguished points. The result is still a frame. Any point in the skeleton of this new frame was in the skeleton of the old because the numbers N_{j_1, j_2} are reduced by removing points of the original frame. Since the j points were not extremities they were followed by points in the old skeleton. Moreover because any two distinguished points are separated the succeeding point is not distinguished. In fact, because there is a point of the spine between any two distinguished points, the succeeding point is also in the new skeleton. Thus the j points are the final points of the gaps in which they lie in the new frame. What we have shown is that the original procedure yields a frame with $a_i - s + j$ elements and any j separated points in its spine. Let $S_{u,j}$ be the number of ways of selecting j separated points from a cyclic set with u elements.

The sum (3) is thus equal to

$$\sum_{s=0}^{c'} \sum_{j=0}^{s} \binom{a_i}{s-j} (-1)^j N^e_{a_i - 2s} S_{a_i - 2(s-j), j} p^{mb_i/a_i (s-j)}$$

[30] because the number of subsets with $a_i - (s-j)$ elements is $\binom{a_i}{s-j}$. This sum may be rewritten as

$$\sum_{j=0}^{c'} \sum_{j=0}^{s} \binom{a_i}{j} (-1)^{s-j} N^e_{a_i - 2s} S_{a_i - 2j, s-j} p^{\frac{mb_i}{a_i} j}.$$

What we have to show is that

$$\sum_{s=j}^{c'} (-1)^{s-j} N^e_{a_i - 2s} S_{a_i - 2j, s-j} = \begin{cases} 0 & j \neq c' \\ 1 & j = c' \end{cases}. \tag{4}$$

Taking the sum over $s - j$ rather than j so that $j \to 0$, $c' \to c - j$, $a_i \to a_i - 2j$ we see that this follows from §5 of the appendix.

I next treat the case that $\overline{G}_i(\mathbf{Q}_p) = \mathrm{GL}(2, F_{\mathfrak{p}_i})$. This time one of the conditions on $x = \{L_j\}$ is that

$$L_j = d\sigma^{n_i}(L_{j+n_i}).$$

Since d is a scalar this implies

$$\overline{L}_j = \sigma^{n_i}(\overline{L}_j).$$

Recalling that the Bruhat-Tits building is a tree, one deduces immediately that at least one of the points $\overline{L}_j$, and hence one of the L_j, is defined over $F_{\mathfrak{p}_i}$. The skeleton is the set of all

j for which L_j is defined over $F_{\mathfrak{p}_i}$. If j lies in the skeleton and j' is the first index in the skeleton to follow it then either $j' = j+1$ or $\overline{L}_{j'} = \overline{L}_j$, for the path of shortest length joining $\overline{L}_{j'}$ to $\overline{L}_j$ must lie in the path defined by x.

I first treat the case that γ is a scalar. I introduce a set U' as before. In each possible skeleton X we fix an index $j_0 = j_0(X)$. A point x will lie in U' if $\gamma \in T_x^m$ and L_{j_0} is the standard lattice V if $j_0 = j_0(X)$, for X the skeleton of x. The desired value of $\varphi_i^m(\gamma)$ is given on p. 11 of the [**31**] old letter. Since

$$\frac{\operatorname{meas} G'(\gamma, F_{\mathfrak{p}_i})\backslash G'(F_{\mathfrak{p}_i})}{\operatorname{meas} G'(O_{\mathfrak{p}_i})} = \frac{q-1}{\operatorname{meas} \mathrm{GL}(2, O_{\mathfrak{p}_i})}$$

if $q = p^{n_i}$ we see that we have to show that $|U'| = 0$ unless

$$\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \qquad |\alpha| = |p^{mb_i/2n_i}|$$

and a_i is even when it should be

$$\binom{a_i}{a_i/2} p^{mb_i/2} - (q-1) \sum_{j=0}^{a_i/2-1} \binom{a_i}{j} p^{mb_i/2}.$$

The analysis proceeds pretty much as before so I do not repeat it in detail. The only difference is that the path associated to the skeleton is a closed path in a tree in which every vertex lies on $q+1$ edges and that the initial edge of a digit cannot be rational over $F_{\mathfrak{p}_i}$ so that there are $q+1$ forbidden directions. Thus (3) is replaced by

$$|U'| = \sum \left(1 - \frac{q}{p^{mb_i/a_i}}\right)^r p^{mb_i/a_i} N_{c'-s}$$

if $N_{c'-s}$ is the number of closed paths with a given origin in a tree in which every vertex lies on $q+1$ edges, a_i is now $2c'$. Proceeding as before we see that the right side equals

$$\sum_{s=0}^{c'} \sum_{j=0}^{s} \binom{a_i}{j} (-1)^{s-j} q^{s-j} N_{c'-s} S_{a_i-2j,s-j} p^{\frac{mb_i}{a_i} j}.$$

What we must do is show that [**32**]

$$\sum_{s=j}^{c'} (-1)^{s-j} q^{s-j} N_{c'-s} S_{a_i-2j,s-j} = \begin{cases} 1 & j = c' \\ -(q-1) & j < c' \end{cases}$$

This is done in §6 of the appendix.

Now suppose γ is not a scalar. The expected value of $\varphi_i^m(\gamma)$ for γ elliptic is given on p. 10 of the old letter. It is 0 unless $p^{-ma_i/\ell_i} \operatorname{Nm} \gamma$ is a unit. Set $q_0 = -1$ if γ is unramified and $q_0 = 0$ if γ is ramified. I observe that

$$\frac{\operatorname{meas} G'(\gamma, F_{\mathfrak{p}_i})\backslash G'(F_{\mathfrak{p}_i})}{\operatorname{meas} G'(O_{\mathfrak{p}_i})} = \frac{-(q_0-1)}{\operatorname{meas} G'(\gamma, O_{\mathfrak{p}_i})}.$$

Here $G'(O_{\mathfrak{p}_i})$ is the group of units in the quaternion algebra over $F_{\mathfrak{p}_i}$ and $G'(\gamma, O_{\mathfrak{p}_i})$ is the group of units in the field generated by γ over $F_{\mathfrak{p}_i}$. The notation is unfortunately rather inadequate. The expression (1) on p. 10 of the letter is then

$$\frac{-(q_0-1)}{\operatorname{meas} G'(\gamma, O_{\mathfrak{p}_i})} \sum_{0\leqslant j<a_i/2} \binom{a_i}{j} p^{\frac{mb_i}{a_i}j}. \tag{5}$$

The expressions (α) and (β) are simply

$$\frac{1}{\operatorname{meas} \mathrm{GL}(2, O_{\mathfrak{p}_i})} \int_{\overline{G}_i(\gamma,\mathbf{Q}_p)\backslash \overline{G}_i(\mathbf{Q}_p)} \eta(g^{-1}\gamma g)\, dg \tag{6}$$

if η is the characteristic function of

$$p^{ma_i/2\ell_i}\mathrm{GL}(2, O_{\mathfrak{p}_i}) = p^{ma_i/2\ell_i}\overline{G}_i(\mathbf{Z}_p).$$

Recall that

$$\overline{G}_i(\mathbf{Q}_p) = \mathrm{GL}(2, F_{\mathfrak{p}_i}).$$

[**33**] The expression (6) is equal to

$$\sum \frac{1}{\operatorname{meas}\Big(\overline{G}_i(\gamma, \mathbf{Q}_p) \cap h\overline{G}_i(\mathbf{Z}_p)h^{-1}\Big)}. \tag{7}$$

The sum is over those double cosets $\overline{G}_i(\gamma, \mathbf{Q}_p)h\overline{G}_i(\mathbf{Z}_p)$ for which

$$h^{-1}\gamma h \in p^{ma_i/2\ell_i}\overline{G}_i(\mathbf{Z}_p).$$

Thus $\varphi_i^m(\gamma)$ is to be (5) if a_i is odd and the sum of (5) and

$$\binom{a_i}{a_i/2} p^{mb_i/2} \tag{8}$$

times (7) if a_i is even provided of course $p^{-ma_i/\ell_i}\operatorname{Nm}\gamma$ is a unit. (In the letter I wrote $p^{ma_i/2}$ instead of $p^{mb_i/2}$. This was a slip.)

If γ is hyperbolic at $\mathfrak{p}_i$ we set $q_0 = 1$. Then (ii) of the old letter is simply demanding that $\varphi_1^m(\gamma)$ be the sum of (5) and of (7) times (8). Of course (5) will be 0. Moreover (7) will be 0 unless

$$\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad |\alpha| = |\beta| = |p^{ma_i/2\ell_i}|.$$

The two periodicity conditions are

$$\sigma^{mn_i/\ell_i}(L_j) = d^{-m/\ell_i}\gamma^{n/\ell_i}(L_j)$$
$$L_{j-\ell_i} = \gamma^u d^v \sigma^{vn_i}(L_j).$$

They guarantee that $\sigma_1^m(\gamma)$ will be 0 unless $|\operatorname{Nm}\gamma| = |p^{ma_i/\ell_i}|$ and both eigenvalues of γ have the same absolute value.

We may introduce the skeleton as before. If j belongs to the skeleton consider the stabilizer G_j of L_j in $\mathrm{GL}(2, F_{\mathfrak{p}_i})_\gamma$, the stabilizer of γ in $\mathrm{GL}(2, F_{\mathfrak{p}_i})$. G_j is the group of units in some order of $F_{\mathfrak{p}_i}(\gamma)$ which contains [**34**] $O_{\mathfrak{p}_i}$, so it makes sense to speak of the set of j for which G_j is maximal. This set I call the reduced skeleton. If j_1 lies in the reduced skeleton and j_2 is the first point in the reduced skeleton to follow it then either $\overline{L}_{j_1} = \overline{L}_{j_2}$ or $j_2 = j_1 + 1$ because $G_{j_1} = G_{j_2}$ stabilizes the path of shortest length joining $\overline{L}_{j_1}$ and $\overline{L}_{j_2}$.

I want to associate an "apartment" to γ in the Bruhat-Tits building of $\mathrm{SL}(2, F_{\mathfrak{p}_i})$. $\mathrm{GL}(2, F_{\mathfrak{p}_i})_\gamma$ may be identified with the multiplicative group of $F_{\mathfrak{p}_i}(\gamma)$. Replacing γ by a conjugate if necessary I may assume that $\mathrm{GL}(2, O_{\mathfrak{p}_i})_\gamma$ is the group of units in $F_{\mathfrak{p}_i}(\gamma)$. Let

$\overline{V}$ be the point in the Bruhat-Tits building associated to the standard lattice. Taking all translates of $\overline{V}$ by $\mathrm{GL}(2, F_{\mathfrak{p}_i})_\gamma$ together with the edges joining any two neighboring points of this set I obtain what I shall call here the "apartment" belonging to γ. It is easily seen to be a connected tree in which every vertex lies in $q_0 + 1$ edges.

The stabilizer of the lattice hV, $h \in \mathrm{GL}(2, F_{\mathfrak{p}_i})$ in $\mathrm{GL}(2, F_{\mathfrak{p}_i})_\gamma$ is determined by the double coset $\mathrm{GL}(2, F_{\mathfrak{p}_i})_\gamma h \mathrm{GL}(2, O_{\mathfrak{p}_i})$ and, as one verifies by analyzing the three possible cases separately, the stabilizer in turn determines the double coset.

Choose a set of double coset representatives h. Set

$$H(h) = \mathrm{GL}(2, F_{\mathfrak{p}_i})_\gamma \cap h\mathrm{GL}(2, O_{\mathfrak{p}_i})h^{-1}.$$

For each possible reduced skeleton X choose an index $j_0 = j_0(X)$ in it. Let $U'(h)$ be the set of all $x \sim \{L_i\}$ for which $\gamma \in T^m_x$, for which the stabilizer of L_j in $\mathrm{GL}(2, F_{\mathfrak{p}_i})_\gamma$ is $H(h)$ if j lies in the reduced skeleton, and for which $L_{j_0} = V$ if $j_0 = j_0(X)$ and X is the reduced skeleton of x. **[35]** Using an argument introduced earlier one sees that

$$\varphi^m_1(\gamma) = \sum_h \frac{\left|U'(h)\right|}{\operatorname{meas} U(h)}.$$

As before when calculating the right side I may suppose $b_i = n_i$.

Suppose 1 is not in the double coset containing h but j, $j+1$ are two consecutive points in the reduced skeleton. Then $\overline{L}_j$, $\overline{L}_{j+1}$ are a distance 1 apart and are the same positive distance from the apartment. This is impossible. Thus for such an h the path associated to the reduced skeleton must be a point. Thus the path which is of length a_i is closed so a_i is even. Analyzing the possible paths as before and observing that the only forbidden direction for the initial edge of a digit is that which moves into the apartment, we see that

$$\left|U'(h)\right| = \binom{a_i}{a_i/2} p^{mb_i/2}$$

for the total number of progressive edges is $s = a_i/2$. This result is to be compared with (7) and (8).

All we have left to do is show that $\left|U'(1)\right|$ is equal to

$$-(q_0 - 1) \sum_{0 \leqslant j < a_i/2} \binom{a_i}{j} p^{\frac{mb_i}{a_i} j} + \begin{cases} 0 & a_i \text{ odd} \\ \binom{a_i}{a_i/2} p^{mb_i/2} & a_i \text{ even} \end{cases}.$$

Let e be the distance between $\overline{L}_j$ and $\gamma^u d^v(\overline{L}_j)$ for j in the reduced skeleton. e is 0 if $q_0 = \pm 1$ and is 0 or 1 if $q_0 = 0$. Since

$$|\det \gamma^u d^v| = |p^{a_i}|$$

we must have $a_i = 2c' + e$. Also e must be 0 if $q_0 = \pm 1$. Anyhow the analysis which led to (3) shows that **[36]**

$$\left|U'(1)\right| = \sum \left(1 - \frac{q_0}{p^{mb_i/a_i}}\right)^r p^{\frac{mb_i}{a_i} s} N^e_f(q_0)$$

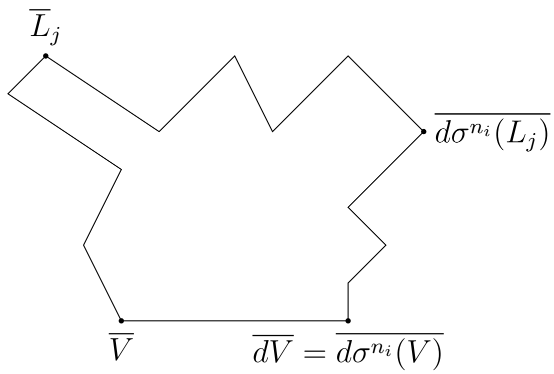
if $N^e_f(q_0)$ is the number of paths of length f in the apartment associated to γ which join two points a distance e apart. The sum is again over all subsets with at least $c' + e$ elements modulo a_i. That the sum has the correct value is clear for $q_0 = 0$. For $q_0 = 1$ this follows from §5 of the appendix as before. For $q_0 = -1$, one uses the argument used when discussing scalar matrices, except that q_0 replaces q, to deduce it from §6 of the appendix.

Finally I must consider the case that $\overline{G}_i(\mathbf{Q}_p)$ is $G'(F_{\mathfrak{p}_i})$ the multiplicative group of the quaternion algebra over $F_{\mathfrak{p}_i}$. Then d is a scalar times

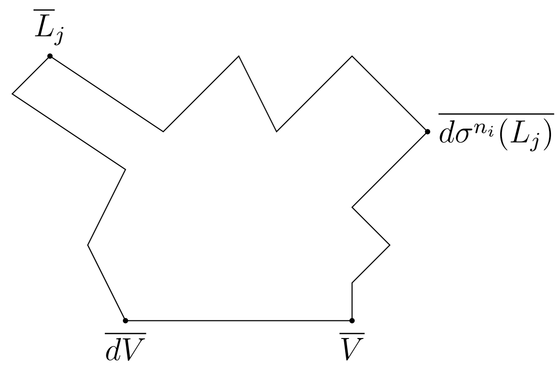
$$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

Now I single out the tree T_1 in the Bruhat-Tits building for $\mathrm{SL}(2, \mathfrak{k})$ formed by $\overline{V}$, the standard lattice, $\overline{dV}$, and the edge joining them. Given $x \sim \{L_j\}$ choose L_j so that the distance from $\overline{L}_j$ and T_1 is minimal

Picture



or



We conclude that $\overline{L}_j \in T_1$.

We now define the skeleton as the set of j for which $\overline{L}_j \in T_1$. If γ is a scalar we define U' about as before by demanding that $L_{j_0} = V$ for a certain j_0. If q_0 is taken to be 0, the by now familiar analysis shows that [**37**]

$$|U'| = \sum (1 - q_0/p^{mb_i/a_i})^r p^{\frac{mb_i}{a_i} s} N_f^e(0)$$

provided U' is not empty. Since $N_f^e(0) = 1$,

$$|U'| = \sum_{0 \leqslant j < a_i/2} \binom{a_i}{j} p^{\frac{mb_i}{a_i} j}$$

as required.

If γ is regular and ramified then $\overline{G}_i(\gamma, \mathbf{Q}_p) = G'(\gamma, F_{\mathfrak{p}_i})$ acts transitively on T_1 so we can define U' in the same way and find the same value for its cardinality. Since

$$\frac{\operatorname{meas} G'(\gamma, F_{\mathfrak{p}_i}) \backslash G'(F_{\mathfrak{p}_i})}{\operatorname{meas} G'(O_{\mathfrak{p}_i})} = \frac{1}{\operatorname{meas} G'(\gamma, O_{\mathfrak{p}_i})} \tag{9}$$

this gives the value for φ_i^m demanded on p. 10 of the old letter.

If γ is regular and unramified, the left side of (9) is equal to

$$\frac{2}{\operatorname{meas} G'(\gamma, O_{\mathfrak{p}_i})}.$$

However, this time $G'(\gamma, O_{\mathfrak{p}_i})$ does not act transitively on $\left\{ L \,\middle|\, \overline{L} = \overline{V} \text{ or } \overline{dV} \right\}$, so we can only demand that L_{j_0} is either V or dV. This accounts for the factor 2. Otherwise we proceed as before.

That does it. It has not been very elegant, but it was not intended to be. All I wanted to do was verify that the method works to give new results. The elegance will I hope come later.

All the best,
R. Langlands

Appendix

I shall verify here various combinatorial facts used in the letter.

1. Consider the infinite tree T_0 formed from the non-negative integral points in the line and the edges joining them. Let $\ell_0 = 1$ and if $n > 0$ let ℓ_n be the number of paths of length $2n$ beginning and ending at 0 but not passing through 0 anywhere else. Set

$$\varphi(t) = \sum_{n=1}^{\infty} \ell_n t^n.$$

Then

$$\ell_n = \frac{(2n-2)!}{(n-1)!n!}$$

and

$$\varphi(t) = \frac{1 = \sqrt{1-4t}}{2}.$$

Only the second statement needs to be verified. Clearly

$$\ell_{n+1} = \sum_{r=1}^{\infty} \sum_{\substack{k_1+\cdots+k_r=n \\ k_i>0}} \ell_{k_1}\ell_{k_2}\cdots\ell_{k_r}.$$

Thus

$$\sum_{r=1}^{\infty} \varphi(t)^r = \sum_{n=2}^{\infty} \ell_n t^{n-1} = \frac{\varphi(t)}{t} - 1$$

so

$$\frac{\varphi(t)}{1-\varphi(t)} = \frac{\varphi(t)}{t} - 1$$

or

$$\varphi(t)^2 = \varphi(t) + t = 0.$$

This gives the result. [**2**]

2. Consider a connected tree on which every vertex lies on $q+1$ edges, $q \geqslant -1$. The number N_n of paths of length $2n$ beginning and ending at a given point is the coefficient of t^n in

$$\frac{1-(q+1)\varphi(qt)}{1-(q+1)^2 t}.$$

To each path we are going to associate a path of length $2n$ in T_0 beginning and ending at 0. This path is defined as follows. If $0 \leqslant k \leqslant 2n$ let the path of minimal length in the tree which connects the kth vertex of the given path to the initial point have n_k edges. Since $n_{k+1} = n_k \pm 1$ and $n_0 = n_{2n} = 0$ the sequence $\{n_k\}$ defines a path in T_0. This new path can be decomposed into a sequence of r paths of length $2k_1, \dots, 2k_r$ with $k_1 + k_2 + \cdots + k_r = n$, each of which begins and ends at 0 but does not otherwise pass through 0. $k_1, \dots, k_r$ will be called type of the original path.

There are clearly

$$\left(1 + \frac{1}{q}\right)^r q^n \ell_{k_1} \cdots \ell_{k_r}$$

paths of type $k_1, \dots, k_r$. Thus

$$\sum_{n=0}^{\infty} N_n t^n = \sum_{r=0}^{\infty} \left(1+\frac{1}{q}\right)^r \varphi(qt)^r = \frac{1}{1-\left(1+\frac{1}{q}\right)\varphi(qt)}.$$

But

$$\begin{aligned}\left\{1-\left(1+\frac{1}{q}\right)\varphi(qt)\right\}\{1-(q+1)\varphi(qt)\} &= 1-\left(q+2+\frac{1}{q}\right)\varphi(qt)\\ &\quad+\left(1+\frac{1}{q}\right)(q+1)\varphi^2(qt)\\ &= 1+\frac{(q+1)^2}{q}\{\varphi^2(qt)-\varphi(qt)\}\\ &= 1-(q+1)^2 t.\end{aligned}$$

[**3**]

3.

$$N_n = \sum_{j=0}^{n} \frac{(2n)!(2n-2j+1)}{j!(2n-j+1)!} q^j.$$

Denote the right side by M_n and consider

$$\left(1-(q+1)^2 t\right)\left(\sum_{n=0}^{\infty} M_n t^n\right).$$

We have to show that it is equal to $1-(q+1)\varphi(qt)$. We compare the coefficients of t^n. For $n=0$ they are clearly equal. Otherwise we have to show that

$$(q+1)q^n \frac{(2n-2)!}{n!(n-1)!} = (q+1)^2 M_{n-1} - M_n.$$

The coefficient of q^{n+1} on the right is

$$\frac{(2n-2)!}{(n-1)!n!}$$

and the coefficient of q^n is

$$\begin{aligned}&\frac{(2n-2)!3}{(n-2)!(n+1)!}+\frac{2(2n-2)!}{(n-1)!n!}-\frac{(2n)!}{n!(n+1)!}\\ &\qquad = -\frac{(2n-2)!}{n!(n+1)!}\{3(n-1)n+2n(n+1)-2n(2n-1)\} = \frac{(2n-2)!}{(n-1)!n!}\end{aligned}$$

We have to show that the coefficients of all other powers of q on the right are 0. The coefficient of q for $n>1$ is

$$2+\frac{(2n-2)!(2n-3)}{(2n-2)!}-\frac{(2n)!(2n-1)}{(2n)!} = 2+(2n-3)-(2n-1) = 0.$$

If $2 \leqslant j < n$ the coefficient of q^j is

$$\frac{(2n-2)!(2n-2j-1)}{j!(2n-j-1)!} + \frac{2(2n-2)!(2n-2j+1)}{(j-1)!(2n-j)!} + \frac{(2n-2)!(2n-2j+3)}{(j+1)!(2n-j+1)!} - \frac{(2n)!(2n-2j+1)}{j!(2n-j+1)!}.$$

Removing the common factor

$$\frac{(2n-2)!}{j!(2n-j+1)}.$$

We obtain [**4**]

$$(2n-2j-1)(2n-j)(2n-j+1) + 2j(2n-2j+1)(2n-j+1) + (2n-2j+3)j(j-1) - 2n(2n-1)(2n-2j+1)$$

which upon simplification turns out to be 0.

4. Let T be the tree obtained from the real line by taking the integers as vertices. Suppose $e \geqslant 0$ and $f = 2f' + e$. Let N_f^e be the number of paths of length f in T joining 0 to e. Then

$$\sum_{f'=0}^{\infty} N_f^{e_t} f' = \frac{\varphi(t)^e}{(1-4t)^{1/2}}.$$

This is seen by observing that the left side is

$$\left(\sum_{n=0}^{\infty} N_n t^n\right)\left(\sum_{n=0}^{\infty} \ell_n t^n\right)^e$$

if N_n is calculated for $q = 1$. But then

$$\sum_{n=0}^{\infty} N_n t^n = \frac{1}{(1-4t)^{1/2}}.$$

5. Consider a cyclic set with u elements. Suppose $2j \leqslant u$. Let $S_{u,j}$ be the number of ways of selecting j objects from the set so that any two are separated.

Example. $S_{6,2} = 9$

$$\left.\begin{array}{l} \times\circ\times\circ\circ\circ\text{: cyclic permutations give 6 possibilities} \\ \times\circ\circ\times\circ\circ\text{: cyclic permutations give 3 possibilities} \end{array}\right\} \text{Total} = 9.$$

In general

$$S_{u,j} = \frac{u(u-j-1)!}{j!(u-2j)!}.$$

This is clear if $j = 0$ or 1. Thus we only need to prove it for $u \geqslant 4$. Let $T_{u,j}$ be the number of ways of choosing j objects from u objects, labeled $1, \ldots, u$ and *not* taken cyclically, so that any two of the j objects are separated. [**5**]

Example. $T_{4,2} = 3$

$$\times\circ\circ\times \quad \circ\times\circ\times \quad \circ\times\circ\times.$$

Clearly

$$S_{u,j} = \frac{u}{j} T_{u-3,j-1}$$

so we show that

$$T_{u,j} = \frac{(u-j+1)!}{j!(u-2j+1)!}.$$

Suppose we have chosen j points from the set $\{1, \ldots, u\}$ so that they are separated. Consider the first $j-1$ of these points. If they are $i_1, \ldots, i_{j-1}$ remove $i_1+1, \ldots, i_{j-1}+1$ from the set $\{1, \ldots, u\}$ to obtain a totally ordered set with $u-j+1$ elements together with a subset of j elements.

Example.

$$\times\circ\circ\times\circ\times \to \times\circ\times\times.$$

To reverse the procedure when we start from a totally ordered set of $u-j+1$ elements, j of which are marked $(\times)$ and the rest of which are unmarked, we just insert an extra circle immediately to the right of the first $j-1$ crosses. Thus

$$\times\times\times\times \to \times\circ\times\circ\times\circ\times.$$

The formula for $T_{u,j}$ follows immediately.

6. Suppose $a = 2c + e$, then

$$(*) \qquad \sum_{s=0}^{c} (-1)^s N^e_{a-2s} S_{a,s} = \begin{cases} 0 & 0 \neq c \\ 1 & 0 = c \end{cases}.$$

[**6**] All we need to do is interpret the sum correctly. Suppose we are given a linearly ordered points with s distinguished points i_α among them, so that any two distinguished points are separated by undistinguished points. Remove the s distinguished points together with the points $i_\alpha + 1$ immediately following them (cyclically)

$$\circ\times\circ\circ\times\circ\circ\times\circ \to \circ\times\times\circ\times\times\circ\times\times \to \circ \quad \circ \quad \circ.$$

There are $a-2s$ points left. Suppose we have a path of length $a-2s$ joining 0 to e in T. There is an obvious indexing of the edges of this path by the remaining $a-2s$ points. The ith edge is indexed by the ith remaining points. We now extend the path to a path of length a.

If i_α is not the final point we add a segment of length two to the path by starting at the final vertex of the edge indexed by $i_\alpha - 1$, proceeding one step to the right and then one step to the left. We do this for each such i_α. If i_α is the final point we add to the path an initial segment starting at 0, going to -1, and then back to 0.

The sum in which we are interested is obtained by taking the sum over all paths of length a of the sum over s from 0 to c of $(-1)^s$ times the multiplicity with which the path is obtained by the above process. Consider a given path. Let k be the number of segments in the path consisting of moving once to the right and then once back to the left. The contribution to the sum from subsets which do not contain the final element of the set of a elements is

$$\sum_{s=0}^{k} (-1)^s \binom{s}{k} = \begin{cases} 0 & k > 0 \\ 1 & k = 0 \end{cases}.$$

Thus the only path for which this sum is not 0 is the one which starts at 0, moves a distance c to the left, and then moves directly back to e. The other subsets only yield a non-zero result for a path which begins at 0, moves to -1, then back to 0 and continues from there. If $c = 0$

there is [**7**] no such path and the sum $(*)$ is 1. Otherwise the argument just given shows that the contribution to the sum for a given path from subsets containing the final element is 0 unless the path starts at 0, moves to -1, moves back to 0, then directly out to $-(c-1)$, and finally back to 0. The contribution for this path is -1. Thus the sum $(*)$ is 0.

7. For any $q \geqslant -1$ consider

$$(**) \qquad \sum_{s=0}^{n} (-1)^s q^s N_{n-s} S_{2n,s}$$

where N_{n-s} is computed with respect to q. I claim that this is sum is 1 if $n = 0$ and that it is $-(q-1)$ otherwise. Since this is easy to see for $q = -1$, I suppose $q \geqslant 0$.

If $n = 0$ the sum is clearly 1. For $n = 1$ we obtain

$$N_1 - 2qN_0 = (1+q) - 2q = -q + 1.$$

The expression $(**)$ is in fact a polynomial of degree n in q. Let $0 \leqslant r \leqslant n$. The coefficient of q^r is

$$\sum_{s=0}^{r} (-1)^s \frac{2n(2n-s-1)!}{s!(2n-2s)!} \cdot \frac{(2n-2s)!(2n-2r+1)}{(r-s)!(2n-s-r+1)!}.$$

It is easily verified that the sum is 1 if $r = 0$ and -1 if $r = 1$. All we have to do is show that it is 0 for $r \geqslant 2$. We rewrite it as

$$\frac{(2n-2r+1)2n}{r!} \sum_{s=0}^{r} (-1)^s \frac{r!}{s!(r-s)!} (2n-s-1) \cdots (2n-s-r+2).$$

We may ignore the initial factor. The sum in this expression is the value at $t = 1$ of

$$\frac{d^{r-2}}{dt^{r-2}} \left\{ \left(1 - \frac{1}{t}\right)^r t^{2n-1} \right\}$$

and that is 0.

Compiled on May 7, 2024.