

Bonn
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Dear Lang,

It occurred to me after I sent you the last letter that it would be worthwhile to see if the considerations in it agreed with Serre's suggestions about the form of the Γ -factor. I just want to explain now that as far as the matter can be checked formally everything is alright.

I had better explain first in more detail, at least for a group over \mathbf{R} , how one goes about defining the dual group and, in the cases of interest to us, local zeta-functions corresponding to representations of this group.

\tilde{G} will be a semi-simple Lie group over \mathbf{C} with Lie algebra $\tilde{\mathfrak{g}}$ and fixed Cartan subalgebra $\tilde{\mathfrak{h}}$. Fix also an order on the roots and hence a positive Weyl chamber. Choosing a Chevalley basis with root vectors e_α we obtain a real (even rational) structure on $\tilde{\mathfrak{g}}$ and hence on \tilde{G} . Let C_0 be the complex anti-linear involution on $\tilde{\mathfrak{g}}$ with respect to this structure. C_0 determines the split real structure on $\tilde{\mathfrak{g}}$ of \tilde{G} . Any other real structure can be obtained by an involution C which maps $\tilde{\mathfrak{h}}$ into itself. $C = IC_0$ where I is an automorphism of $\tilde{\mathfrak{g}}$. The real structure is inner or outer according as I is inner or outer. Let L_+ be the lattice of weights of $\tilde{\mathfrak{h}}$ —(or $\tilde{\mathfrak{g}}$), L_- the sublattice generated by the roots, and L , where $L_- \subseteq L \subseteq L_+$, the weights of $\tilde{\mathfrak{g}}$ which actually define weights of \tilde{G} . α denotes a root. We have the following scheme:

$$\begin{array}{ccc}
 \alpha & & \hat{\alpha} \\
 \cap & & \cap \\
 L_- & \hat{L}_- = \text{Hom}(L_+, \mathbf{Z}) & \\
 \cap & \cap & \hat{\alpha}(\lambda) \text{ (or } \langle \lambda, \hat{\alpha} \rangle \text{) is} \\
 L & \hat{L} = \text{Hom}(L, \mathbf{Z}) & 2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \text{ if } \lambda \in L_+. \text{ } (\cdot, \cdot) \\
 \cap & \cap & \text{is the Killing form on } \tilde{\mathfrak{h}}. \\
 L_+ & \hat{L}_+ = \text{Hom}(L_-, \mathbf{Z}) &
 \end{array}$$

We may suppose that $\hat{\alpha}$ is simple or positive only when α is. There is a semi-simple Lie algebra $\hat{\mathfrak{g}}$ and a corresponding group \hat{G}_0 and Cartan subalgebra $\hat{\mathfrak{h}}$ for which the roofed objects have the same significance as the unroofed for $\tilde{\mathfrak{g}}$, \tilde{G} , and $\tilde{\mathfrak{h}}$. The Weyl groups of $\hat{\mathfrak{h}}$ in $\hat{\mathfrak{g}}$ and $\tilde{\mathfrak{h}}$ in $\tilde{\mathfrak{g}}$ are isomorphic ($\sigma \leftrightarrow \hat{\sigma}$) in such a way that $\langle \sigma\lambda, \hat{\sigma}\hat{\lambda} \rangle = \langle \lambda, \hat{\lambda} \rangle$. \hat{G}_0 is the connected component of the dual group for any real form G of \tilde{G} . Let G be determined by the involution $C = IC_0$. I is the product of an inner automorphism normalizing $\tilde{\mathfrak{h}}$ and an outer automorphism permuting the positive simple roots of $\tilde{\mathfrak{h}}$. Every such permutation naturally determines a permutation of the positive simple $\hat{\alpha}$ and hence an automorphism of \hat{G}_0 (called the straight extension in Freudenthal-de Vries). We take \hat{G} to be the corresponding

split extension of \widehat{G}_0 so that $[\widehat{G} : \widehat{G}_0] = 2$ and $\widehat{G} = \widehat{G}_0 \cup \omega\widehat{G}_0$ where $\omega^2 = 1$ (I note that two forms which differ by an inner twisting have the same dual group. I also observe that, according to the definitions of my Washington lecture, \widehat{G} is not uniquely determined—if G is inner one is free to take $\widehat{G}_0 = \widehat{G}$ —however the present choice is here convenient.) ω corresponds to an automorphism of the Dynkin diagram and therefore also to a permutation of the highest weights $\widehat{\lambda} \rightarrow \omega\widehat{\lambda}$. Let $\widehat{\sigma}_0$ be an irreducible representation of \widehat{G}_0 with highest weight $\widehat{\lambda}$. Then $\widehat{\sigma}_0(\omega g \omega^{-1})$ has highest weight $\omega\widehat{\lambda}$. If $\widehat{\lambda} = \omega\widehat{\lambda}$ there is an A such that

$$A\widehat{\sigma}_0(g)A^{-1} = \widehat{\sigma}_0(\omega g \omega^{-1}).$$

A takes the vector corresponding to λ to a scalar multiple of itself. We may suppose that this scalar is ± 1 so that $A^2 = 1$. Setting $\widehat{\sigma}(g) = \widehat{\sigma}_0(g)$ and $\widehat{\sigma}(\omega) = A$ we obtain a representation $\widehat{\sigma}$ of \widehat{G} (that with highest weight $\widehat{\lambda}$.) If $\widehat{\lambda} \neq \omega\widehat{\lambda}$ set

$$\widehat{\sigma}(g) = \begin{pmatrix} \widehat{\sigma}_0(g) & 0 \\ 0 & \widehat{\sigma}_0(\omega g \omega^{-1}) \end{pmatrix} \quad g \in \widehat{G}_0,$$

$$\widehat{\sigma}(\omega) = \pm \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

$\widehat{\sigma}$ is irreducible and is said to have the highest weight $\widehat{\lambda}$ (or $\omega\widehat{\lambda}$). These are the only representations of \widehat{G} we need to consider.

We are interested at first in the case that $G = G_{\text{un}}$ is the compact form. Then the effect of I is to take every root into its negative. G_{un} is inner if and only if this can be effected by an element of the Weyl group. When $G = G_{\text{un}}$ then $\widehat{\mu}$ is a root of $\widehat{\sigma}$ only when $-\widehat{\mu}$ is also. Moreover $\omega\widehat{G}_0$ contains an element ϵ which normalizes $\widehat{\mathfrak{h}}$ and sends roots to their negatives. ϵ is determined up to conjugacy by an element of $\exp \widehat{\mathfrak{h}} = \widehat{T}_0$.

I want to describe how in accordance with the suggestions of my Washington lecture the irreducible representations of G_{un} correspond to certain homomorphisms of the Weil group $W_{\mathbf{C}/\mathbf{R}}$ into \widehat{G} . Let $W_{\mathbf{C}/\mathbf{R}} = \mathbf{C}^\times \cup \delta\mathbf{C}^\times$ where $\delta^2 = -1$. I consider only homomorphisms τ which map \mathbf{C}^\times into \widehat{T}_0 and δ into ϵ . If $\widehat{\mu}$ belongs to \widehat{L} and $\xi_{\widehat{\mu}}$ is the corresponding rational character of \widehat{T}_0 there is an $m(\widehat{\mu}) \in \mathbf{Z}$ such that

$$\xi_{\widehat{\mu}}(\tau(z)) = \left(\frac{z}{|z|} \right)^{m(\widehat{\mu})}.$$

Define $2\Lambda \in L$ by $m(\widehat{\mu}) = \langle 2\Lambda, \widehat{\mu} \rangle$ for every $\widehat{\mu}$. Then

$$\xi_{\widehat{\mu}}(\tau(-1)) = (-1)^{\langle 2\Lambda, \widehat{\mu} \rangle} = \xi_{\widehat{\mu}}(\epsilon^2).$$

ϵ^2 does not depend on the choice of ϵ . By some messy calculations involving classification one can show that

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^{\langle 2\rho, \widehat{\mu} \rangle}$$

if 2ρ is the sum of the positive roots of $\widehat{\mathfrak{h}}$. Thus $2\Lambda - 2\rho$ is even for all $\widehat{\mu}$ in \widehat{L} and $\Lambda - \rho$ is in L . Of course any Λ satisfying this condition arises from a suitable τ . The irreducible representation $\pi(\tau)$ of G_{un} corresponding to τ is (this is a definition) the representation with natural parameter Λ . To obtain this representation one chooses w in the Weyl group so that $w\Lambda$ lies in the positive Weyl chamber. The representation with “natural” parameter Λ is that with highest weight $w\Lambda - \rho = w(\Lambda - \rho) + w\rho - \rho$

($w\rho - \rho \in L_- \subseteq L$). Thus $\pi(\tau)$ is only defined if Λ is non-singular. (This fits perfectly well with my Washington lecture since G_{un} is not quasi-split.)

If G is any form with compact Cartan subgroup then G is an inner form of G_{un} , so \widehat{G} is defined as above. In this case the representations of G corresponding to τ are those members of the discrete series for G with a parameter Λ' which equals $w\Lambda$ for some w in the (complex) Weyl group of G . We set (a definition)

$$L(s, \pi(\tau), \widehat{\sigma}) = L(s, \widehat{\sigma} \circ \tau)$$

if $\widehat{\sigma}$ is a complex-analytic representation of \widehat{G} . $L(s, \widehat{\sigma} \circ \tau)$ is the Γ -factor (defined as in my notes on Artin L -functions) for the representation $\widehat{\sigma} \circ \tau$ of $W_{\mathbf{C}/\mathbf{R}}$. We consider those $\widehat{\sigma}$ introduced above. Every time that a non-zero weight $\widehat{\mu}$ occurs in $\widehat{\sigma}$ so does $-\widehat{\mu}$. Thus the representation

$$\text{Ind}(W_{\mathbf{C}/\mathbf{R}}, W_{\mathbf{C}/\mathbf{C}}, \delta_{\langle \widehat{\mu}, 2\Lambda \rangle})$$

with

$$\delta_m(z) = \left(\frac{z}{|z|} \right)^m : \mathbf{C}^\times \rightarrow \mathbf{C}^\times$$

is included in $\widehat{\sigma} \circ \tau$ as often as $\widehat{\mu}$ is a weight of $\widehat{\sigma}$. Therefore each time that $\widehat{\mu}$ occurs one gets a factor

$$2(2\pi)^{-(s+|\langle \widehat{\mu}, \Lambda \rangle|)} \Gamma(s + |\langle \widehat{\mu}, \Lambda \rangle|)$$

in $L(s, \widehat{\sigma} \circ \tau)$. On the spaces corresponding to the weight 0, $\widehat{\sigma} \circ \tau$ kills \mathbf{C}^\times and is a representation of $\mathfrak{G}(\mathbf{C}/\mathbf{R}) = W_{\mathbf{C}/\mathbf{R}}/\mathbf{C}^\times$. Every time that this representation contains the trivial representation we get a factor

$$\pi^{-\frac{1}{2}s} \Gamma\left(\frac{s}{2}\right)$$

and every time that it contains the non-trivial representation we get a factor

$$\pi^{-\frac{1}{2}(s+1)} \Gamma\left(\frac{s+1}{2}\right)$$

Notice that if the weight 0 does not occur in $\widehat{\sigma}_0$ the function $L(s, \pi(\tau), \widehat{\sigma})$ depends only on $\widehat{\sigma}_0$.

We have to apply this to the representation $\widehat{\sigma}_0 = \sigma_0$ introduced in my last letter. According to the vague suggestions of that letter a certain part of the cohomology group in the middle dimension is broken up into d -dimensional subspaces. To compare with Serre's suggestion we have to see how much of this d -dimensional part corresponds to a given $H^{p,q}$. We arrange things so that $\widetilde{\mathfrak{h}}$ corresponds to a compact Cartan subgroup of G and so that, with respect to the order chosen, every non-compact positive root is totally positive. (To fix ideas I have in mind the case that G is simple and defined over \mathbf{Q} . Presumably $\dim \widehat{\sigma}_0 = \dim \widehat{\sigma}$ (with the given $\widehat{\sigma}_0 = \sigma_0$) corresponds to the case that we should think of Shimura's variety \widetilde{V} as defined over \mathbf{Q} and $\dim \widehat{\sigma}_0 = \frac{1}{2} \dim \widehat{\sigma}$ to the case that we should think of \widetilde{V} as defined over a certain imaginary quadratic extension of \mathbf{Q} —see remark in previous letter.)

The tangent space T at $K \in K \backslash G = X$ breaks up into the direct sum of the holomorphic tangent vectors T^+ and the anti-holomorphic T^- . If $t = \frac{1}{2} \text{Real dim } X$

then

$$\bigwedge^t T = \bigoplus_{p+q=t} \bigwedge^p (T^+) \otimes \bigwedge^q (T^-).$$

Let ρ_+ be one-half the sum of the positive non-compact roots (with respect to the given order). This order determines a positive (or for us standard) Weyl chamber W . Take another Weyl chamber W' and let ρ' be one-half the sum of the corresponding positive roots and ρ'_+ one-half the sum of the corresponding non-compact positive roots. Choose w in W_G , the Weyl group of $G_{\mathbf{C}}$, so that $wW = W'$ and let $T_{\rho'}$ be the element of the discrete series corresponding to ρ' . $T_{\rho'}$ depends only on the coset $W_K w$, if $W_K \subseteq W_G$ is the Weyl group of $K_{\mathbf{C}}$. Let p positive non-compact roots with respect to W' be positive with respect to W and let q be negative. Then according to Matsushima's discussion (at least formally) and the remarks at the end of the last letter the representation of K with extreme weight $2\rho'_+$ occurs with multiplicity one in $\bigwedge^p T^+ \otimes \bigwedge^q T^-$ so that the one-dimensional part of the d -dimensional space contributed by $T_{\rho'}$ lies in $H^{p,q}$.

However $\dim \hat{\sigma}_0 = [W_G : W_K] = d$ so that the weights of $\hat{\sigma}_0 (= \sigma_0)$ are $\hat{w}^{-1} \hat{\mu}_0$, w varying over representations of $W_K \backslash W_G$. In particular 0 is not a weight. Finally

$$\langle \hat{w}^{-1} \hat{\mu}_0, 2\rho \rangle = \langle \hat{\mu}_0, w(2\rho) \rangle = \langle \hat{\mu}_0, 2\rho' \rangle = \langle \hat{\mu}_0, 2\rho'_+ \rangle$$

because $\hat{\mu}_0$ annihilates compact roots. Because of the definition of $\hat{\mu}_0$

$$\langle \hat{\mu}_0, 2\rho'_+ \rangle = p - q.$$

Moreover the relevant part of the Hasse-Weil(?) zeta-function should be

$$L\left(s - \frac{t}{2}, T_{\rho}, \hat{\sigma}\right) \quad \text{with } t = \frac{1}{2} \dim X = p + q$$

if $\hat{\sigma}$ is the representation constructed from $\hat{\sigma}_0$ as above. Since

$$-\frac{p+q}{2} + \frac{|p-q|}{2} = -\inf(p, q),$$

we are in perfect agreement with Serre (Sem. Delange-Pisot-Poitou—May 1970). Of course this supposes that the previous parenthetical remarks are O.K. When $\dim \hat{\sigma}_0 = \dim \hat{\sigma}$ so that, under the circumstances mentioned, \tilde{V} is to be thought of as defined over \mathbf{Q} one has also to check the effect of Serre's σ on $H^{\frac{t}{2}, \frac{t}{2}}$ (when t is even). This one can of course do only in those cases where Shimura has actually defined \tilde{V} . I have not yet checked anything.

Yours truly,

Bob Langlands

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