

APPENDIX TO LETTER TO SERGE LANG—DECEMBER 17,
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It is enough to prove the relation

$$\xi_{\hat{\mu}}(\epsilon^2) = (-1)^{\langle 2\rho, \hat{\mu} \rangle}$$

when \tilde{G} , and therefore also \hat{G}_0 is simple. ϵ^2 certainly lies in the Cartan subgroup of \hat{G}_0 corresponding to $\hat{\mathfrak{h}}$. \hat{G} acts as a group of automorphisms of $\hat{\mathfrak{g}}$ (of course not faithfully). Let (\cdot, \cdot) be the Killing form on $\hat{\mathfrak{g}}$. \hat{G} preserves the Killing form. Let $e_{\hat{\alpha}}$ be a root vector. Then

$$(e_{\hat{\alpha}}, \epsilon e_{\hat{\alpha}}) \neq 0$$

and

$$(e_{\hat{\alpha}}, \epsilon e_{\hat{\alpha}}) = (\epsilon e_{\hat{\alpha}}, \epsilon^2 e_{\hat{\alpha}}) = \xi_{\hat{\alpha}}(\epsilon^2)(\epsilon e_{\hat{\alpha}}, e_{\hat{\alpha}}) = \xi_{\hat{\alpha}}(\epsilon^2)(e_{\hat{\alpha}}, \epsilon e_{\hat{\alpha}})$$

because the Killing form is symmetric. Thus $\xi_{\hat{\alpha}} = 1$. Since $\langle \rho, \hat{\alpha} \rangle$ is integral and in fact equal to 1 for simple $\hat{\alpha}$,

$$\xi_{\hat{\alpha}} = (-1)^{\langle 2\rho, \hat{\alpha} \rangle}.$$

In particular the assertion is true in those cases that $\hat{L}_- = \hat{L}_+$. Thus when $\hat{\mathfrak{g}}$ is of type E_8 , F_4 , or G_2 .

In the other cases let $\hat{\beta}$ be the top root (as defined in Freudenthal-de Vries). Then $\omega e_{\hat{\beta}} = \eta e_{\hat{\beta}}$, with $\eta = \pm 1$, and $\omega e_{-\hat{\beta}} = \eta e_{-\hat{\beta}}$. Thus, taking the standard isomorphism of the group corresponding to

$$\text{Span}\{e_{\hat{\beta}}, e_{-\hat{\beta}}, [e_{\hat{\beta}}, e_{-\hat{\beta}}]\}$$

with $\text{SL}(2, \mathbf{C})$, ω corresponds to the trivial automorphism if $\eta = 1$ and to the inner automorphism determined by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

if $\eta = -1$. The reflection corresponding to $\hat{\beta}$ is determined by

$$\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus $\omega\delta\omega^{-1} = \delta$ if $\eta = +1$ and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

if $\eta = -1$. We regard δ as an element of \hat{G}_0 . It takes every root which is not orthogonal to $\hat{\beta}$ to a negative root. The roots orthogonal to $\hat{\beta}$ are linear combinations of the simple roots orthogonal to $\hat{\beta}$.

We consider the various types separately. \hat{H}_0 will be the group corresponding to the algebra generated by root vectors belonging to roots orthogonal to $\hat{\beta}$. \hat{H}_0 is invariant under ω . \hat{H} will be $\hat{H}_0 \cup \omega\hat{H}_0$. If \hat{H}_0 is simple we may suppose our assertion is proved for \hat{H} . We use the tables in Fr.-De Vries.

$\widehat{\beta}$ is $2\omega_1$. ω is again the trivial automorphism. \widehat{H}_0 is now the group obtained by removing ω_1 from the Dynkin diagram. If δ_1 is the element of \widehat{H}_0 taking roots to their negatives we may take $\epsilon = \omega\delta\delta_1$ so that $\epsilon^2 = \delta^2\delta_1^2$. By induction

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^x$$

with

$$\begin{aligned} x &= \langle \widehat{\mu}, \omega_1 \rangle + \langle \widehat{\mu}, (0, 2\ell - 3, \dots, 1) \rangle \\ &= \langle \widehat{\mu}, (1, 2\ell - 3, \dots, 1) \rangle \end{aligned}$$

The difference between 2ρ and $(1, 2\ell - 3, \dots, 1)$ is $(2(\ell - 1), 0, \dots, 0)$. The value of this element of L_+ at any $\widehat{\mu}$ in \widehat{L}_+ is even.

(iv) $D_\ell(=\widehat{\mathfrak{g}})$. Here

$$\rho = (2\ell - 2, 2\ell - 4, \dots, 0).$$

$\widehat{\beta}$ is $\omega_1 + \omega_2$. We take $\ell > 3$. We take \widehat{H}_0 by removing ρ_4 from the Dynkin diagram. We may take

$$e_{\widehat{\beta}} = [e_{\rho_4}, [e_{\rho_5}, [\dots, [e_{\rho_\ell}, [e_{\rho_3}, [\dots, [e_{\rho_{\ell-1}}, [e_{\rho_2}, [e_{\rho_\ell}, e_{\rho_1}]] \dots]]]]]]]$$

Now

$$[e_{\rho_2}, [e_{\rho_\ell}, e_{\rho_1}]] + [e_{\rho_1}, [e_{\rho_2}, e_{\rho_\ell}]] + [e_{\rho_\ell}, [e_{\rho_1}, e_{\rho_2}]] = 0.$$

the last term is 0; so

$$[e_{\rho_2}, [e_{\rho_\ell}, e_{\rho_1}]] = [e_{\rho_1}, [e_{\rho_\ell}, e_{\rho_2}]].$$

If ℓ is even, ω is trivial and η is 1. If ℓ is odd it follows from these calculations that η is also 1, because $\omega e_{\rho_2} = e_{\rho_1}$ and $\omega e_{\rho_1} = e_{\rho_2}$. Let δ_1 be the reflection corresponding to ρ_3 and let δ_2 be that element of the Weyl group of \widehat{H}_0 taking positive roots to negative roots. Take $\epsilon = \omega\delta\delta_1\delta_2$. Then by induction

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^x$$

with

$$x = \langle \widehat{\mu}, \omega_1 + \omega_2 \rangle + \langle \widehat{\mu}, \omega_1 - \omega_2 \rangle + \langle \widehat{\mu}, (0, 0, 2\ell - 6, \dots, 0) \rangle = \langle \widehat{\mu}, (2, 0, 2\ell - 6, \dots, 0) \rangle.$$

The difference between ρ and $(2, 0, 2\ell - 6, \dots, 0)$ is $(2\ell - 4, 2\ell - 4, 0, \dots, 0)$ which takes an even value at every $\widehat{\mu}$. If $\ell = 4$ the induction argument cannot be used. In this case δ_2 is the product of the reflection corresponding to ρ_1 and that corresponding to ρ_2 . Thus

$$\xi_{\widehat{\mu}}(\delta_2^2) = (-1)^y$$

with

$$y = \langle \widehat{\mu}, \omega_3 - \omega_4 \rangle + \langle \widehat{\mu}, \omega_3 + \omega_4 \rangle.$$

This justifies the use of the induction.

E_6 : The toproot is

$$\begin{array}{cccccc} & & & 2 & & \\ & & & & & \\ & & & 1 & 2 & 3 & 2 & 1 \end{array}$$

Thus \widehat{H}_0 is of type D_6 . Let $2\rho_0$ be the sum of the positive roots for this D_6 . We have to show that $\rho - \beta - 2\rho_0$ is a sum of simple roots in which each root enters an even number of times. Using the tables as before, we obtain

$$\begin{array}{ccccccc}
 & & & & & \cdot 49 & \\
 2\rho & \longleftrightarrow & \cdot 27 & \cdot 52 & \cdot 75 & \cdot 96 & \cdot 66 & \cdot 34 \\
 & & & & & \cdot 15 & & \\
 2\rho_0 & \longleftrightarrow & \cdot 10 & \cdot 18 & \cdot 24 & \cdot 28 & \cdot 15 & \cdot 0
 \end{array}$$

Everything checks.

As an extra check let's work out G_2 and F_4 along these lines. Again the Lie algebras and their duals are the same.

F_4 : Toproot is

$$2 \text{ --- } 3 \Rightarrow 4 \text{ --- } 2$$

ω is of course trivial. The diagram is

$$\begin{array}{ccc}
 \begin{array}{cc} 2 & 2 \\ \bullet \text{---} \bullet \\ -1 \end{array} & \Rightarrow & \begin{array}{cc} 1 & 1 \\ \bullet \text{---} \bullet \\ -1/2 \end{array} \\
 & & -1
 \end{array}$$

Thus $(\widehat{\beta}, \widehat{\alpha})$ is given by

$$\begin{array}{ccc}
 \begin{array}{cc} 2 & 2 \\ \bullet \text{---} \bullet \\ 4-3=1 \quad 6-2-4=0 \end{array} & \Rightarrow & \begin{array}{cc} 1 & 1 \\ \bullet \text{---} \bullet \\ 4-1-3=0 \quad 2-4/2=0 \end{array}
 \end{array}$$

\widehat{H}_0 is in this case C_3 and its dual is B_3 .

$$2\rho_0 \longleftrightarrow \begin{array}{cc} \bullet \text{---} \bullet \\ 0 \quad 9 \end{array} \Leftarrow \begin{array}{cc} \bullet \text{---} \bullet \\ 8 \quad 5 \end{array}$$

Note that in passing to the dual the direction of the arrow is reversed.

$$\begin{array}{ccc}
 2\rho & \longleftrightarrow & \begin{array}{cc} 22 & 42 \\ \bullet \text{---} \bullet \end{array} & \Leftarrow & \begin{array}{cc} 30 & 16 \\ \bullet \text{---} \bullet \end{array} \\
 \beta & \longleftrightarrow & \begin{array}{cc} 2 & 3 \\ \bullet \text{---} \bullet \end{array} & \Leftarrow & \begin{array}{cc} 2 & 1 \\ \bullet \text{---} \bullet \end{array}
 \end{array}$$

because $(\beta, \rho_2) = (\beta, \rho_4) = (\beta, \rho_3) = 0$. Note again that arrows are reversed. Note also that there is only one positive root orthogonal to ρ_2, ρ_3 and ρ_4 . In any case it checks.

G_2 : Toproot is

$$2 \Rightarrow 3$$

The diagram is

$$\begin{array}{ccc}
 3 & & 1 \\
 \bullet & \Leftarrow & \bullet \\
 & & -3/2
 \end{array}$$

$(\widehat{\beta}, \widehat{\alpha})$ is given by

$$6 - \overset{\bullet}{\frac{9}{2}} = \overset{\bullet}{\frac{3}{2}} \quad \Rightarrow \quad 3 - \overset{\bullet}{\frac{3}{2}} \cdot 2 = 0$$

Thus, in this case,

$$\begin{array}{ccc} 2\rho_0 & \longleftrightarrow & \overset{\bullet}{0} \Leftarrow \overset{\bullet}{1} \\ 2\rho & \longleftrightarrow & \overset{\bullet}{10} \Leftarrow \overset{\bullet}{6} \end{array}$$

and, since β annihilates the root on the right

$$\beta \longleftrightarrow 2 \Leftarrow 1.$$

Everything checks in this case also.

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