

June 2, 1965

Dear Professor Ono,

I am writing to ask if it is known that on the basis of the theory of Eisenstein series it is possible to give a formula for the volume of the fundamental domain of a discrete group associated to a Chevalley group defined over a number field. I shall give you the formula and sketch a proof of it below but first some preliminary remarks.

Suppose  $\mathfrak{G}_K$  is a split semi-simple Lie algebra over the number field  $K$ ; suppose  $\mathfrak{h}_K$  is a splitting Cartan subalgebra;  $\theta$  is an automorphism of  $\mathfrak{G}_K$  with  $\theta(X) = -X$  for  $X \in \mathfrak{h}_K$ ; and suppose for each root  $\alpha$   $X_\alpha$  is a root vector and  $\theta(X_\alpha) = -X_{-\alpha}$  and  $\alpha(H_\alpha) = 2$  if  $H_\alpha = [X_\alpha, X_{-\alpha}]$ . Suppose also that  $\mathfrak{G}_K$  is actually a Lie algebra of linear transformations of a finite-dimensional vector space  $V_K$  over  $K$ . There is of course a connected algebraic group  $G$  of linear transformations on  $V$  associated to  $\mathfrak{G}_K$ .

Let  $\alpha_1, \dots, \alpha_p$  be the simple positive roots of  $\mathfrak{h}_K$ . Let  $A_1, \dots, A_p$  be  $p$  fractional ideals in  $K$  and if  $\alpha = \sum_{i=1}^p m_i \alpha_i$  is a root set  $A_\alpha = \prod_{i=1}^p A_i^{m_i}$ . If  $M$  is an  $O$ -module in  $V_K$  which contains a basis of  $V_K$  then  $M$  is said to be admissible (with respect to  $\mathfrak{h}_K, \{X_\alpha\}, \{A_\alpha\}$ ) if, for all  $\alpha$  and all  $n \geq 0$

$$\frac{1}{n!} A_\alpha^n (M X_\alpha^n) \subseteq M$$

and

$$M = \sum_{\lambda \in L} M \cap V_K(\lambda)$$

$L$  is the set of integral linear functions and, for  $\lambda \in L$ ,  $V_K(\lambda)$  is the associated weight space. Then the group of all  $\gamma$  in  $G_K$  such that  $M\gamma = M$  is independent of the choice of the admissible  $m$  adelic  $M$ . (cf. Sem. Bourbaki, #219). Let  $G_O$  be this group.

If  $\mathfrak{G}_p$  is the direct product of  $p$  copies of  $\mathfrak{G}$ , the ideal of class group of  $K$  and  $\lambda \in L$ , let  $\pi(\lambda)$  be the homomorphism of  $\mathfrak{G}_p$  into  $\mathfrak{G}$  defined by

$$\pi(\lambda)(\mathfrak{a}_1, \dots, \mathfrak{a}_p) = \prod_{i=1}^p \mathfrak{a}_i^{\lambda(H_{\alpha_i})}$$

and let  $\mathfrak{G}(V)$  be the intersection over those  $\lambda$  such that  $V_K(\lambda) \neq \{0\}$  of the kernel of  $\pi(\lambda)$ . Then one can show that the number of conjugacy classes of Borel subgroups, defined over  $K$ , with respect to  $G_O$  is  $[\mathfrak{G}_p : \mathfrak{G}(V)]$ . Indeed there is a simple correspondence between the conjugacy classes of Borel subgroups and the cosets of  $\mathfrak{G}(V)$  in  $\mathfrak{G}_p$ . It is probably possible to show that the number  $c(G)$  introduced by Borel in his paper on adèle groups is  $[\mathfrak{G}(V) : 1]$  in the present case but I have not done this.

Let  $G_r$  be the set of real points on the connected group associated to  $\mathfrak{G}_K$  considered as an algebra over  $\mathbf{Q}$  acting on  $V_K$  considered as a vector space over  $\mathbf{Q}$ .

For each  $\alpha$  let  $\{X_\alpha^i \mid 1 \leq i \leq m\}$  be a basis for  $A_\alpha$ ; let  $\{X^i\}$  be a basis for  $O$  over  $\mathbf{Z}$  and let  $H_i, 1 \leq i \leq p$ , be a basis over  $\mathbf{Z}$  of  $\{H \in \mathfrak{h}_K \mid \lambda(H) \in \mathbf{Z} \text{ if } V_K(\lambda) \neq \{0\}\}$ . Then

$$\left\{ X_\alpha^i X_\alpha \mid \alpha \text{ a root, } 1 \leq i \leq n \right\} \cup \left\{ X^i H_j \mid 1 \leq i \leq n, 1 \leq j \leq p \right\}$$

is a basis for the Lie algebra of  $G_r$ . Let  $dg_r$  be the measure on  $G_r$  obtained from the form dual to this basis, then

$$(*) \quad \int_{G_O \backslash G_r} dg_r = \frac{\epsilon}{[\mathfrak{G}(V) : 1]} \prod_{i=1}^p f(a_i, K)$$

with  $\epsilon$  equal to the order of the fundamental group of  $G$  and  $a_1, \dots, a_p$  as in formula (8) of your recent paper.  $f(a_i, K)$  is the value of the zeta function of  $K$  at  $a_i$ .

In the proof which is sketched below the references are to some mimeographed notes of mine on Eisenstein series, a copy of which I am sending to you.

If  $N_r$  is the connected group whose Lie algebra is spanned by

$$\{X'_\alpha X_\alpha \mid \alpha > 0, 1 \leq i \leq n\},$$

with a measure  $dn$  dual to this basis, if  $A_r$  is the connected group in  $G_r$  whose Lie algebra is spanned by  $\{X^i H_j\}$  with a measure  $da$  dual to this basis, and if  $K = \{k \in G_r \mid \theta(k) = k\}$  with a measure  $dk$  so that  $\int_K dk = 1$  let  $d_1g$  be such that

$$\int_G \phi(g) d_1g = \int_{N_r} \int_{A_r} \int_K dn_r da_r dk \omega^2(a) \phi(nak).$$

Here  $d(an_r a^{-1}) = \omega^{-2}(a) dn_r$  is the definition of  $\omega(a)$ . Let

$$\Phi(s, K) = (2^{-r_2} \pi^{-n/2} \Delta^{1/2})^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \delta(s, K).$$

Using an integral formula due to Gindikin and Karpelevich one sees that  $(*)$  is equivalent to

$$(**) \quad \int_{\Gamma \backslash G} d_1g = \frac{c}{[\mathfrak{G}(V) : 1]} \prod_{i=1}^p \Phi(a_i, K) \left( \prod_{a>0} \text{Nm } A_\alpha \right)$$

Let  $P_1, \dots, P_r$  be a set of representatives for the conjugacy classes of Borel subgroups and let  $P$  be the Borel subgroup with the Lie algebra spanned by  $\{X_\alpha \mid \alpha > 0\} \cup \{H_j\}$ . If  $g_r \in G_r$  and  $g = (g_1, \dots, g_n)$  where  $n = [K : \mathbf{Q}]$ , and  $g_1 \in G_{\mathbf{C}}$  let  $g_i = a_i n_i u_i$ ,  $a_i \in A_{\mathbf{R}}$ ,  $n_i \in N_{\mathbf{C}}$ ,  $u_i \in$  unitary restriction of  $G_{\mathbf{C}}$ . If  $a_i = \exp H_i$  and  $\lambda$  is a linear function on  $\mathfrak{a}_{\mathbf{C}}$  set

$$F(g, H) = \exp \left( \lambda \left( \sum_{i=1}^n H_i \right) + \rho \left( \sum_{i=1}^n H_i \right) \right).$$

$\rho$  is one-half the sum of the positive roots. For each  $j$  one can choose ideals  $A_1^j, \dots, A_p^j$  in a special way and an element  $h_j$  in a special way so that  $h_j^{-1} P h_j = P_j$ . If  $\lambda + \rho = \sum \mu_i \alpha_i$  set

$$F(g, \lambda, P_j) = \prod_{i=1}^p (\text{Nm } A_i)^{M_i} \prod_{i=1}^p (\text{Nm } A_i^j)^{-H_i} F(h_j g, P)$$

If  $\phi_j(\cdot)$  is a measurable function on  $N_j(G_O \cap P_j) \backslash G/K$ , then  $\phi_j(\cdot)$  can be represented as an integral

$$\phi_j(g) = \frac{1}{(2\pi)^p} \int_{\operatorname{Re} \lambda = \lambda_0} \Phi_i(\lambda) F(g, \lambda, P_j) (d\lambda).$$

where  $\Phi_i(\lambda)$  is an entire function. It is easy to see that if  $\widehat{\phi}_j(g) = \sum_{\Delta_j \backslash \Gamma} \phi_j(\gamma g)$  then

$$\begin{aligned} (\widehat{\phi}_j(\cdot), 1) &= \int_{\Gamma \backslash G} \widehat{\phi}_j(g) d_1 g = R^p \Phi_j(\rho) \\ R &= \frac{\Delta^{-1/2} 2^n \pi^{r_2} M}{w d_{r+1}} \end{aligned}$$

( $M$  is defined as in Landau's book on Algebraic Numbers.) Let  $\widehat{\phi} = \sum_{j=1}^p \widehat{\phi}_j$  and let  $\Pi$  be the orthogonal projection on the space of constant functions. If  $\widehat{\psi} = \sum \widehat{\psi}_j$  then

$$(1, 1)(\Pi \widehat{\phi}, \Pi \widehat{\psi}) = (\widehat{\phi}, 1) \overline{(\widehat{\psi}, 1)}$$

Since

$$\int_{\Gamma \backslash G} d_1 g = (1, 1)$$

we can compute the volume of the fundamental domain if we can find a formula for  $(\Pi \widehat{\phi}, \Pi \widehat{\psi})$ . Such a formula can be obtained from the main theorem of my notes on Eisenstein series together with a little computation. The result is

$$\frac{1}{c} (hR)^p \frac{1}{\prod_{i=1}^p \Phi(a_i, K)} \frac{(\sum_i \Phi_i(\rho)) \overline{(\sum_j \Phi_j(\rho))}}{[\mathfrak{G}_p : \mathfrak{G}(V)]} \left\{ \prod_{\alpha > 0} \operatorname{Nm} A_\alpha \right\} R^p$$

which one sees leads to the desired result.

If the formula (\*) is of any interest we will perhaps have an opportunity to discuss the proof in more detail at Boulder if you are attending the Institute there this summer.

Yours truly,

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Compiled on May 1, 2026.