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Linearization of webs of codimension one and maximum rank

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§ 1. The theorem

Let R^n be the n-dimensional real number space with the coordinates $x_1, \cdots, x_n$. In a neighborhood U of R^n consider d families of hypersurfaces defined by

$$u_i(x_1, \cdots, x_n)=\text{const}, \qquad 1\leqq i\leqq d. \tag{1}$$

We suppose the functions u_i to be smooth, with grad $u_i\neq 0$. Each family is called a *foliation*, of codimension one, and its hypersurfaces the *leaves*. The totality of the d foliations is called a *d-web*. We require further that the tangent hyperplanes to the d leaves through a point of U be in general position. Clearly the ith foliation will remain unchanged if u_i is replaced by a function $v_i(u_i)$ with $v_i'\neq 0$.

An equation of the form

$$\sum_i f_i(u_i)du_i=0 \tag{2}$$

is called an *abelian equation*. Such an equation is invariant under the changes $u_i\to v_i(u_i)$ $(v_i'\neq 0)$ and is therefore a property of the web. For example, when $n=2$, $d=3$, an abelian equation can, by proper choices of the u_i, be written

$$u_1+u_2+u_3=0. \tag{3}$$

It follows that the web can be mapped locally into three families of parallel straight lines. Not every 3-web in the plane has this property. Those which do are called *hexagonal*. They have interesting geometrical properties; cf. [1].

The number of linearly independent abelian equations (over the reals) is called the *rank*. It was proved in [4] that the rank r of a d-web of codimension one in R^n has an upper bound depending only on d and n:

$$r\leqq\pi(d, n)=\frac{1}{2(n-1)}(d-1-s)(d-n+s), \tag{4}$$

where s is defined, uniquely, by the conditions

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(5) $$s \equiv -d+1, \bmod n-1, \qquad 0 \leqq s \leqq n-2.$$

It is an elementary fact that $\pi(d, n)$ is an integer; this will also follow from § 3.

We have the following theorem:

Theorem 1. *Consider a d-web of codimension one in R^n of maximum rank $\pi(d, n)$. Suppose that $n \geqq 3$, $d \geqq 2n$. Then the web is linearizable, that is, there is a coordinate system in a neighborhood relative to which all the leaves are hyperplanes.*

This theorem was proved by G. Bol for $n=3$ [1]. It is not true for $n=2$, nor for $d<2n$, $d \neq n+1$; cf. [1], [3].

§ 2. Sketch of proof

Suppose the abelian equations be

(6) $$\sum_i f_i^\lambda(u_i) du_i = 0, \qquad 1 \leqq \lambda \leqq \pi = \pi(d, n),$$

which are linearly independent. Let

(7) $$Z_i(x) = [f_i^1(u_i), \cdots, f_i^\pi(u_i)], \qquad 1 \leqq i \leqq d,\ x \in U,$$

be the point in an auxiliary projective space $P^{\pi-1}$ of dimension $\pi-1$ having the right-hand side as its homogeneous coordinates. For the sake of brevity we will not distinguish in the following the homogeneous coordinate vector and the point it represents in the projective space. Equation (6) can be written

(6a) $$\sum_i Z_i \otimes du_i = 0.$$

For $1 \leqq \alpha \leqq n$ we multiply (6a) by

$$du_1 \wedge \cdots \wedge du_{\alpha-1} \wedge du_{\alpha+1} \wedge \cdots \wedge du_n.$$

By the general position requirement of the web we have

$$du_1 \wedge \cdots \wedge du_n \neq 0.$$

It follows that

(8) $$Z_\alpha = \sum_s p_\alpha^s Z_s, \qquad n+1 \leqq s \leqq d.$$

Hence among the Z_i there are at most $d-n$ linearly independent ones. The assumption of maximum rank implies that exactly $d-n$ of Z_i are linearly independent. Let

(9) $$P^{d-n-1}(x) = \{Z_1(x), \cdots, Z_d(x)\} \subset P^{\pi-1}$$

be the space spanned by $Z_i(x)$. By sending x to $P^{d-n-1}(x)$ we have a mapping

(10) $$U \rightarrow Gr(d-n-1, \pi-1),$$

where the right-hand side is the Grassmann manifold of all linear spaces of dimension $d-n-1$ in $P^{\pi-1}$. The mapping (10) will be called the *Poincaré mapping*. It was used by Poincaré in his work on double surfaces of translation and θ-divisors on abelian varieties.

Using the Poincaré mapping we can prove the theorem in the case $d=2n$ immediately. We have $\pi(2n, n)=n+1$ and $P^{\pi-1}=P^n$. Hence (10) becomes

(10a) $$U \rightarrow \mathrm{Gr}\,(n-1, n)=P^{*n}$$

and is a mapping between spaces of the same dimension. In fact, it sends a point of U into a hyperplane of P^n and a leaf of the web into a point of P^n, as seen from (7). The linearization theorem follows by duality in P^n. From now on we suppose $d>2n$.

Substituting (8) into (6a) and using the fact that Z_s are linearly independent, we get

(11) $$\sum_\alpha p_\alpha^s du_\alpha + du_s = 0.$$

Let PT_x^*, $x \in U$, be the projectivized cotangent space to R^n at x. Then du_i ($\neq 0$) defines a point in PT_x^*. We will call it the *normal* to the ith leaf or simply the *ith web normal*. Equation (11) gives the relation between the d web normals.

Differentiating (8), we get

(12) $$dZ_\alpha \equiv \sum_s p_\alpha^s dZ_s, \qquad \mathrm{mod}\ Z_i,$$

or

(12a) $$Z'_\alpha \otimes du_\alpha \equiv \sum_s p_\alpha^s Z'_s \otimes du_s, \qquad \mathrm{mod}\ Z_i,$$

where

(13) $$Z'_i = dZ_i / du_i.$$

By our general position requirement we have

$$du_\alpha \wedge du_{n+1} \wedge \cdots \wedge du_{2n-1} \neq 0.$$

We multiply (12a) by

$$du_\alpha \wedge du_{n+1} \wedge \cdots \wedge \widehat{du_t} \wedge \cdots \wedge du_{2n-1} \ (du_t \text{ omitted}),$$
$$t=\alpha,\ n+1, \cdots, 2n-1.$$

It follows that

$$p^t_\alpha Z'_t \equiv 0, \qquad \text{mod } Z_i, Z'_{2n}, \cdots, Z'_d,$$

and the same is true of Z'_t itself. Hence $Z_1, \cdots, Z_d, Z'_1, \cdots, Z'_d$ span a space of dimension $\leqq 2d-3n$. The maximum rank hypothesis implies that this maximum dimension is attained. We can therefore set

(14) $$\{Z_i(x), Z'_i(x)\} = P^{2d-3n}(x).$$

Geometrically $Z_i(x)$ describes a curve in $P^{\pi-1}$. $P^{d-n-1}(x)$ is the space spanned by their corresponding points and $P^{2d-3n}(x)$ is the space spanned by the corresponding tangent lines.

Substituting (11) into (12a), we get

$$Z'_\alpha \otimes du_\alpha + \sum_{\beta, s} p^s_\alpha p^s_\beta Z'_s \otimes du_\beta \equiv 0, \qquad \text{mod } Z_i,\ 1 \leqq \alpha, \beta \leqq n,$$

which gives

(15) $$\delta_{\alpha\beta} Z'_\alpha + \sum_s p^s_\alpha p^s_\beta Z'_s \equiv 0, \qquad \text{mod } Z_i$$

and

(15a) $$\sum_s p^s_\alpha p^s_\beta Z'_s \equiv 0, \qquad \text{mod } Z_i,\ \alpha \neq \beta.$$

These are $\frac{1}{2}n(n-1)$ relations. Our maximum rank hypothesis requires that they determine only $n-1$ of the Z'_s, the Z'_α being determined by (15) with $\beta=\alpha$. Hence among the coefficents $p^s_\alpha p^s_\beta$, $\alpha \neq \beta$ of (15a) there are

$$\tfrac{1}{2}n(n-1)-(n-1)=\tfrac{1}{2}(n-1)(n-2)$$

linearly independent relations. Every such relation is of the form

(16) $$\sum_{\alpha, \beta, \alpha \neq \beta} a_{\alpha\beta} p^s_\alpha p^s_\beta = 0, \qquad a_{\alpha\beta} = a_{\beta\alpha}.$$

In T^*_x we can take du_α as a basis. A covector is of the form $\sum_\alpha q_\alpha du_\alpha$ and has the coordinates q_α relative to this basis. An equation of the form

(17) $$\sum_{\alpha, \beta} a_{\alpha\beta} q_\alpha q_\beta = 0, \qquad a_{\alpha\beta} = a_{\beta\alpha}$$

defines a hyperquadric in PT^*_x. We proved above that the d web normals lie on $\infty^{1/2(n-1)(n-2)}$ hyperquadrics. Under our assumption $d > 2n$ they lie on a uniquely determined rational normal curve common to these hyperquadrics. We will denote the curve by D_x. Our maximum rank web leads to the following geometric structure in R^n: *Given in a neighborhood U of R^n a d-web of maximum rank $\pi(d, n)$, there* *is defined a field of rational normal curves D_x in PT_x^*, $x \in U$, such that the d web normals at x lie on D_x.*

It can be proved that, as a consequence of the equation (6a), the d points $Z_i(x) \in P^{d-n-1}(x)$ also lie on a rational normal curve $E(x)$. Moreover, D_x and $E(x)$ are in a projective correspondence under which du^i and $Z_i(x)$ correspond and the cross ratio of four points is preserved.

We pursue further the study of the Poincaré mapping. A tangent direction through x is a point of PT_x and corresponds by duality to a hyperplane of PT_x^*. The latter meets D_x in $n-1$ points, which correspond to $n-1$ points, or an $(n-2)$-dimensional chord, on $E(x)$. This correspondence is reciprocal. Furthermore, to a curve $x(t)$ in U corresponds in $P^{\pi-1}$ a family $E(t) \subset P^{d-n-1}(t)$ with $n-1$ distinguished points on $E(t)$. This can be pictured as a ruled variety $P^{d-n-1}(t)$ with $n-1$ directrix curves transversal to the generators.

The fundamental fact concerns the curves in U to which correspond ruled varieties whose $n-1$ directrix curves reduce to points. These curves are the integral curves of a certain system of ordinary differential equations of the second order. They are the *paths of a projective structure* or a normal projective connection in U; cf. [2]. The situation is perhaps more clearly illustrated by the table:

$U \subset R^n$	$P^{\pi-1}$
point x	$E(x) \subset P^{d-n-1}(x)$
$du_i \in D_x \subset PT_x^*$	$Z_i(x) \in E(x)$, $D_x \barwedge E(x)$
direction $\in PT_x$	$(n-2)$-dim chord of $E(x)$
curve	ruled variety of P^{d-n-1} with $n-1$ directrix curves
path	ruled variety with $n-1$ directrix points
totally geodesic hypersurface	$E(x)$ through a point of Castelnuovo surface

The last line needs explanation. It would seem that all the $E(x)$, $x \in U$, describe an $(n+1)$-dimensional subvariety

$$V = \bigcup_{x \in U} E(x) \tag{18}$$

of $P^{\pi-1}$. But this is not so. For through a point $p_0 \in V$ there are ∞^{n-1} $E(x)$. This can be seen as follows: Let $E_0 \ni p_0$. Another E through p_0 is determined by picking $n-2$ points $p_1, \cdots, p_{n-2}$ on E_0 such that E contains $p_0, \cdots, p_{n-2}$; there are ∞^1 E's through these p's. It follows that V is a two-dimensional surface, to be called the *Castelnuovo surface*. To all the E through a point of V correspond the points of a hypersurface in U. The latter is totally geodesic in the sense that any two points on it, which are sufficiently close, can be joined by path lying completely on it. Among these totally geodesic hypersurfaces are the leaves of the web. Thus the leaves of the d foliations are now imbedded in a two-parameter family of totally geodesic hypersurfaces of a generalized projective geometry.

It remains to prove that the latter is locally flat, i.e., there exists a local coordinate system relative to which the paths are straight lines. This follows from the existence of the ∞^2 totally geodesic hypersurfaces. More precisely, we have the theorem: *Suppose in U there be a normal projective connection and a field of rational normal curves* $D_x \subset PT_x^*$, $x \in U$, *such that tangent to every hyperplane corresponding to a point of* D_x *there is a totally geodesic hypersurface. Then the projective connection is flat.*

The details of the proof of Theorem 1 can be found in [5].

Once the linearization theorem 1 is proved, the complete local description of the web is given by the theorem [6]:

Theorem 2. *Consider a d-web of codimension one in a neighborhood U of* R^n *whose leaves are hyperplanes and which satisfies an abelian equation* (2) *with* $f_i(u_i) \neq 0$. *Then the leaves belong to an algebraic curve of degree d in the dual projective space.*

We remark that for Theorem 2 to be valid only one abelian equation is needed. Moreover, it holds for $n \geqq 2$, $d \geqq n+1$.

§3. Castelnuovo's bound

Our problem is closely related to one studied by Castelnuovo [3]: In the complex projective space P^n consider a non-degenerate algebraic curve C of degree d ("non-degenerate" means that C does not lie in any P^{n-1}). To determine the maximum genus of C. Castelnuovo proved that the genus g of C has the bound

$$g \leqq K\left\{d - \frac{K+1}{2}(n-1) - 1\right\}, \qquad K = \left[\frac{d-1}{n-1}\right], \tag{19}$$

and that the bound is attained.

We wish to verify that the right-hand side of (19) is equal to $\pi(d, n)$. In fact, the definition of K is

$$\frac{d-1}{n-1} = K + \frac{s'}{n-1}, \qquad 0 \leqq s' \leqq n-2,$$

which gives

$$s' \equiv d-1, \qquad \mod n-1.$$

Substituting K into (19), we get

$$g \leqq \frac{1}{2(n-1)}(d-1-s')(d-n+s').$$

The right-hand side is seen to be $\pi(d, n)$, if we notice

$$\begin{aligned} s' &= n-1-s, \qquad 0 < s \leqq n-2, \\ s' &= 0, \qquad\qquad\quad 0 = s. \end{aligned}$$

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