An Interesting 0-Cycle

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1 Introduction

Let X be a smooth algebraic curve of genus g and $L \to X$ a line bundle of degree n. On the surface $Y = X \times X$, following a construction of Carel Faber and Rahul Pandharipande we define a (rational equivalence class of) 0-cycle

$$z_L \in CH^2(Y)$$

as follows: Let D be a divisor on X with $[D] = L$. Then we set

$$z_L = D \times D - nD_\Delta$$

where D_Δ is the divisor D on the diagonal in $X \times X$. It is easy to see that

$$\begin{cases} \deg z_L = 0 \\ \mathrm{Alb}_Y(z_L) = 0 \end{cases}$$

where $\mathrm{Alb}_Y(z_L)$ is the image of z_L in the Albanese variety of Y. We are interested in the question: Is

(1.1) $$z_L \equiv 0 \ ,$$

where $\equiv$ denotes rational equivalence of 0-cycles? This depends only on the line bundle $[D]$, and it is easy to see that (1.1) is true if

$$D = np, \qquad p \in X \ .$$

Additionally, according to the conjectures of Bloch and Beilinson z_L should be torsion in $CH^2(Y)$ if (X, L) is defined over a number field. We shall show that:

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Theorem 1: *If X is a smooth curve of genus $g \geqq 2$ and $L \to X$ is a general line bundle, then*

$$z_L \neq 0 \ \ in \ \ CH^2(X) \ .$$

In fact, the proof will show that z_L is non-torsion in $CH^2(X)$.

Theorem 2: *For $L = K_X$ the canonical bundle, if $g \geqq 4$ and X is the generic then*

$$z_{K_X} \neq 0 \ \ in \ \ CH^2(Y) \ .$$

In Theorem 2 we note that (1.1) holds if X is hyperelliptic, and therefore if $g = 2$. Carel Faber and Rahul Pandharipande have proved that (1.1) also is valid if $g = 3$, and therefore our result is sharp. This paper settles a problem raised by them in their study of the tautological subring of the cohomology and Chow rings of the moduli space of curves as to whether the cycle z_{K_X} on the product of a curve with itself is always rationally equivalent to 0. Interestingly, the invariant we use to show that z_{K_X} is not rationally equivalent to 0 (or even torsion in the Chow group) on a general curve of genus ≥ 4 makes use of an infinitesimal computation on moduli space.

Theorem 1 is relatively elementary and, as pointed out to us by the referee, may be proved by an extension of Mumford's original argument using, e.g., the one given in the introduction to [9]. We have chosen to present the proof given below as it illustrates in a transparent way the idea behind the more complicated computations in Theorem 2. As a general observation, in higher dimensions it is generally difficult to decide if a given 0-cycle is or is not rationally equivalent to zero. Most of the "non-rationally equivalent to zero" results seem to apply to generic situations (e.g. [9], [5]). To us a main interest in the problem is that in Theorem 2, even though X is general the divisor K_X is particular, so that z_{K_X} is somewhere between general and special.

Our method of proof is Hodge-theoretic. For curves the classical Abel-Jacobi mapping

(1.2) $$\mathrm{AJ}_X : \mathrm{Div}^0(X) \to J(X)$$

gives necessary and sufficient Hodge-theoretic conditions that a divisor of degree zero be rationally equivalent to zero. For 0-cycles on a surface, no satisfactory analogue of (1.2) has yet been found[1]. Moreover, the theorems stated above can only hold generically, which

[1] In [M. Green and P. Griffiths, "Hodge-theoretic invariants for algebraic cycles", to appear in IMRN] the authors will propose an analogue for 0-cycles on a surface which will (modulo torsion) capture rational equivalence *if* (and this is a big "if") one assumes the conjectures of Bloch and Beilinson.

suggests the use of variational methods. Associated to a family $\mathcal{D} = \{D_s \in \mathrm{Div}^0(X_s)\}_{s\in S}$ of divisors on a family of smooth curves there is defined a *normal function* $\nu_{\mathcal{D}}$ where

$$\nu_{\mathcal{D}}(s) = \mathrm{AJ}_{X_s}(D_s) \in J(X_s)\ .$$

Although $\nu_{\mathcal{D}}(s)$ is defined transcendentally its associated *infinitesimal invariant* $\delta\nu_{\mathcal{D}}$ may be defined algebraically. Moreover,

$$\delta\nu_{\mathcal{D}} \neq 0 \Rightarrow \nu_{\mathcal{D}}(s) \neq 0 \ \text{ for general } s,$$

and thus $\mathrm{AJ}_{X_s}(D_s) \neq 0$ in $J(X_s) = CH^1(X_s)$ for general s. In fact, since $\delta\nu_{n\mathcal{D}} = n\delta\nu_{\mathcal{D}}$ has values in a vector bundle we have that $\mathrm{AJ}_{X_S}(D_s)$ is non-torsion for general s.

Suppose now that we have a family $\mathcal{Z}$ of 0-cycles z_s on a family of algebraic surfaces Y_s and that

$$\begin{cases} \deg z_s = 0 \\ \mathrm{Alb}_{Y_s}(z_s) = 0\ . \end{cases}$$

Then although we are not yet able to define the analogue $\nu_{\mathcal{Z}}$ of $\nu_{\mathcal{D}}$, we can define the analogue $\delta\nu_{\mathcal{Z}}$ of $\delta\nu_{\mathcal{D}}$ with the property that

$$\delta\nu_{\mathcal{Z}} \neq 0 \Rightarrow z_s \neq 0 \ \text{ in } \ CH^2(Y_s) \ \text{ for general } Y_s.$$

In fact, as above we have that z_s is non-torsion in $CH^2(Y_s)$ for general s. Our construction of $\delta\nu_{\mathcal{Z}}$ is a direct extension of earlier work of [3], [5], [9] and [10]. Their work, and the later work by [1], is applied to families of cycles on hypersurfaces in projective space; there the explicit calculation of $\delta\nu_{\mathcal{Z}}$ is reduced to polynomial algebra. Our main contribution is to introduce some new calculational methods when a polynomial description of the infinitesimal variation of Hodge structure is not available. Although, as noted above, the computation that

$$\delta\nu_L \neq 0$$

in Theorem 1 turns out to be relatively straightforward, the proof that

$$\delta\nu_{K_X} \neq 0$$

in Theorem 2 turns out to be somwehat subtle. The basic idea is to use the Schiffer variation associated to a point $p \in X$, which intuitively may be thought of as a variation that changes the complex structure on X by a "δ-function" at p. This allows us to localize the computation at two distinct points $p, q \in X$, and the condition $g \geqq 4$ enters via the requirement that the tangent lines to the canonical curve at p and q not intersect.

To carry this computation out we have written everything out explicitly in local coordinates — essentially, we need an expression for the relative diagonal in a family $\{X_x \times X_s\}_{s\in S}$ of products of curves. Just preceeding the calculation at the end of section 4(a) we have given a heurestic argument that leaves little doubt that the desired result is true *up to a scale factor which needs to be non-zero.* As suggested by the referee it would be far more satisfactory to replace the calculation by a more conceptual argument using functorial properties of the diagonal, but we have not been able to find such an argument.

The most satisfactory approach could well be to give the computation in the completely intrinsic form described in the problem stated at the end of section 4(c).

The organization of this paper is as follows: In Section 2 we will calculate the infinitesimal invariant for families of line bundles of degree zero over a family of algebraic curves. Although not strictly necessary for the logical development of our story, this case served us as an important "toy model" for the more subtle situations given by the theorems stated above. In Section 3 we will give the proof of Theorem 1; this argument is based on establishing a geometric formula for $\delta\nu_L$. In Section 4 we will first set up the framework for the proof of Theorem 2 and give an heuristic argument for the main calculation. There we also formulate an interesting general problem that for 0-cycles on surfaces would probably give the most satisfactory expression of the relationship between $\delta\nu_{\mathcal{Z}}$ and K-theory. Then in the remainder of Section 4 we will give the complete proof of the theorem. Finally, in the Appendix we will give the definition of the infinitesimal invariants used in this paper. In fact, those utilized in our work here are only part of a general sequence of invariants that one may expect to be of further use in similar geometric problems.

2 Computation of the infinitesimal invariant in the curve case

In this section we will study the infinitesimal invariant of a normal function associated to a family $L_s \to X_s$ of line bundles of degree zero over a family of algebraic curves. The main objective will be to express this invariant in terms of the Kodaira-Spencer class associated to the variation of $L_s \to X_s$ and the variation of Hodge structure associated to the X_s's.

(a) More formally, we consider the situation

(2.1)
$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & \mathcal{X} \\ & & \big\downarrow{\scriptstyle \pi} \\ & & S \end{array}$$

where $\mathcal{X} \xrightarrow{\pi} S$ is a smooth family of smooth algebraic curves of genus $g \geqq 1$ and $\mathcal{L} \to \mathcal{X}$ is a line bundle of degree zero on each fibre. We shall work with complex varieties in the algebraic setting (Zariski topology, etc.). Our considerations will be local in the base space S, which we may assume to be an affine variety. We shall sometimes write $\mathcal{X} \xrightarrow{\pi} S$ as $\{X_s\}_{s \in S}$ where $X_s = \pi^{-1}(s)$, and shall write the fibre of (2.1) over $s \in S$ as

$$L_s \to X_s \ .$$

Finally, we shall denote simply by $L \to X$ the fibre at a generic point s_0 of S and set

$$\begin{cases} T = T_{s_0} S \\ V = H^0(\Omega^1_X) \ . \end{cases}$$

Associated to (2.1) is a normal function $\nu_{\mathcal{L}}$ with infinitesimal invariant $\delta\nu_{\mathcal{L}}$ (cf. the Appendix); $\delta\nu_{\mathcal{L}}$ is a section over S of the sheaf

$$\Omega^1_S \otimes R^1_\pi \mathcal{O}_{\mathcal{X}} / \nabla R^0_\pi \Omega^1_{\mathcal{X}/S}$$

where

$$\nabla : R^0_\pi \Omega^1_{\mathcal{X}/S} \to \Omega^1_S \otimes R^1_\pi \mathcal{O}_{\mathcal{X}} \tag{2.2}$$

is the map induced by the Gauss-Manin connection. Here, and throughout, for any smooth variety Z we write Ω^1_Z for $\Omega^1_{Z/\mathbb{C}}$. The map (2.2) is algebraic; i.e., it is an $\mathcal{O}_S$-linear map between vector bundles, and shrinking S we may assume that it has constant rank. Evaluation of $\delta\nu_{\mathcal{L}}$ at a generic point gives, using the notations introduced above,

$$\delta\nu_L \in T^* \otimes V^* / \nabla V \ . \tag{2.3}$$

Using duality we may write this as

$$\delta\nu_L \in \{\ker\{T \otimes V \to V^*\}\}^* \ . \tag{2.3*}$$

Our objective is to give an expression for $\delta\nu_L$ in terms of the Kodaira-Spencer mapping

$$T \to H^1(\Sigma_L)$$

where Σ_L is the sheaf of linear, 1^{st}-order differential operators on sections of $L \to X$. Before doing this we mention the following tautological

Example: We let $S = J(X)$ be the Jacobian variety of X and

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & X \times J(X) \\ & & \downarrow \\ & & J(X) \end{array}$$

the universal (Poincaré) line bundle. Then

$$\begin{cases} T = H^1(\mathcal{O}_X) = V^* \\ \nabla = 0 \end{cases}$$

and

$$\delta\nu_L \in V \otimes V^*$$

may be easily seen to be the identity (this will be proved by computation in Section 3 below).

Assuming this computation we may draw the following conclusion: Let $S = \mathrm{Pic}^0(\mathcal{X}_g/\mathcal{M}_g)$ be the set of pairs (X, L) where $L \to X$ is a line bundle of degree zero over a smooth curve of genus g. Over S there is a natural normal function $\nu_{\mathcal{L}}$ and we have the

Corollary: $\delta\nu_{\mathcal{L}} \neq 0$.

The reason is that if we fix X then $\nu_{\mathcal{L}}$ restricts to the normal function given in the example above. The geometric conclusion that for $L \to X$ a general line bundle of degree zero we have $L \neq 0$ in $\mathrm{Pic}^0(X)$ is obvious; the point is that the method of proof will extend to the situations described in the introduction.

(2.4) **Remark:** If instead of $J(X) = \mathrm{Pic}^0(X)$ we let $S = \mathrm{Pic}^n(X)$ and

$$\mathcal{L}_n \to X \times \mathrm{Pic}^n(X)$$

be the universal line bundle of degree n, then there is no canonically associated normal function. Choosing $p \in X$ and setting

$$L_s = L_{n,s} - n[p]$$

we do get a family $L_s \to X$ of degree zero line bundles with associated normal function whose infinitesimal invariant is independent of p, and is in fact also given by the identity as in the above example. We may think of $L_{n,s}$ as the "principal part" of L_s; the np is subtracted off to get $\deg L_s = 0$.

The reason this is relevant to the main topic of this paper is the following: For $L \to X$ of degree n and D a divisor with $[D] = L$ the 0-cycle

$$\tilde{z}_L = D \times D$$

on $Y = X \times X$ is of degree n^2. We may subtract off $n^2 p \times p$ to make $\tilde{z}_L$ have degree zero, but then it does not map to zero in $\mathrm{Alb}(Y)$. The easiest way to obtain a 0-cycle z_L with "principal part" $\tilde{z}_L$ and with

$$\begin{cases} \deg z_L = 0 \\ \mathrm{Alb}_Y(z_L) = 0 \end{cases}$$

is to set

$$z_L = \tilde{z}_L - nD_\Delta \ .$$

Now let $L \to X$ vary in $\mathrm{Pic}^n(X)$ to obtain a family $z_L(s)$ of 0-cycles on Y with associated infinitesimal invariant whose value at a general point is (cf. the Appendix)

$$\delta\nu_{z_L} \in \Lambda^2 T^* \otimes \Lambda^2 V^* \ .$$

One may suspect that, as in the curve case, $\delta\nu_{z_L}$ depends only on the principal part of z_L and since $T \cong H^1(\mathcal{O}_X) = V^*$ it is certainly suggested that again

$$\delta\nu_{z_L} = \text{ "identity" } \in \Lambda^2 V \otimes \Lambda^2 V^* \ .$$

This in fact turns out to be the case and implies Theorem 1 by analogy to the corollary above. This argument will be carried out in detail in Section 3 below.

(b) We will now turn to an explicit computation. For a line bundle $L \to X$ a 1^{st}-order linear differential operator on sections of L is given by a $\mathbb{C}$-linear map

$$\mathcal{O}(L) \xrightarrow{D} \mathcal{O}(L)$$

satisfying locally

$$D(f\lambda) = v(f)\lambda + fD\lambda$$

where $f \in \mathcal{O}_X$, $\lambda \in \mathcal{O}(L)$ and $v \in \Theta_X = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X)$ is a vector field. If $v = 0$ so that D is $\mathcal{O}_X$-linear, then

$$D\lambda = g\lambda$$

where $g \in \mathcal{O}_X$. The 1st-order linear differential operators form a sheaf Σ_L, and from the preceding remark we see that there is a natural exact sequence

$$0 \to \mathcal{O}_X \to \Sigma_L \to \Theta_X \to 0 \ . \tag{2.5}$$

Given the situation (2.1) there are Kodaira-Spencer maps

$$T_s S \to H^1(\Sigma_{L_s}) \tag{2.6}$$

that measure the infinitesimal variation of the family $\{L_s \to X_s\}_{s\in S}$ (this does not require that $\deg L_s = 0$). The cohomology sequence of (2.5) has the natural interpretation

$$\begin{array}{ccccc} H^1(\mathcal{O}_X) & \to & H^1(\Sigma_L) & \to & H^1(\Theta_X) \\ \| & & \| & & \| \\ \left\{\begin{array}{c}\text{tangent space to}\\ \text{moduli of } (X,L)\\ \text{with } X \text{ fixed}\end{array}\right\} & & \left\{\begin{array}{c}\text{tangent space to}\\ \text{moduli of}\\ \text{pairs } (X,L)\end{array}\right\} & & \left\{\begin{array}{c}\text{tangent space}\\ \text{of moduli}\\ \text{of } X\text{'s}\end{array}\right\}. \end{array}$$

The obstruction to splitting (2.5) is given by the Atiyah-Chern class

$$c_1(L) \in H^1(\Omega^1_X) \ .$$

Thus, if $\deg L = 0$ the sequence (2.5) splits

$$\Sigma_L \cong \mathcal{O}_X \oplus \Theta_X \ , \tag{2.7}$$

and any two splittings differ by an element of

$$H^0(\mathrm{Hom}(\Theta_X, \mathcal{O}_X)) = H^0(\Omega^1_X) \ .$$

We write the splitting (2.7) as

$$\sigma = (\sigma', \sigma'')$$

where $\sigma \in \Sigma_L$, $\sigma' \in \mathcal{O}_X$ and $\sigma'' \in \Theta_X$. Going to a general point of S, the map (2.6) and splitting (2.7) give

$$\begin{array}{ccc} T_s S & \longrightarrow & H^1(\Sigma_{L_s}) \\ & \searrow & \downarrow \\ & & H^1(\mathcal{O}_X) \end{array}$$

i.e., an element

$$\tau \in T^* \otimes H^1(\mathcal{O}_X) \ .$$

From the above remark, changing the splitting (2.7) changes τ by

$$\tau \to \tau + \rho(\omega)$$

where $\omega \in H^0(\Omega^1_X)$ and ρ is the composite

$$T \to H^1(\Theta_X) \to \mathrm{Hom}\left(H^0(\Omega^1_X), H^1(\mathcal{O}_X)\right)$$

of the usual Kodaira-Spencer map together with the cup product. Since

$$\rho(\omega) = \nabla\omega$$

we have a well-defined element

$$\begin{array}{c}[\tau] \in T^* \otimes H^1(\mathcal{O}_X)/\nabla H^0(\Omega^1_X) \\ \| \\ T^* \otimes V^*/\nabla V \ .\end{array}$$

Theorem: $[\tau] = \delta\nu_L$.

In one sense this result is obvious: What else could $[\tau]$ be? However, to prove it we shall give a calculation that — although not difficult — will establish the notations and set the stage for the more subtle calculations needed for the proofs of Theorems 1 and 2.

Proof: In the following we may work analytically using local holomorphic coordinates or algebraically using local uniformizing paramaters; we shall do the former.

We consider the situation

$$\begin{array}{c}\mathcal{X} \\ \downarrow\pi \\ S\end{array} \tag{2.8}$$

locally over a neighborhood U of $s_0 \in S$ in which there are local coordinates $s = (s_1, \ldots, s_N)$ with $s_0 = 0$. By *data* for the situation (2.8) we shall mean

$$\{U_\alpha, z_\alpha, f_{\alpha\beta}(z_\beta, s)\}$$

where U_α is a covering of $\mathcal{X}$ with $\pi(U_\alpha) = U$, $(z_\alpha, s_1, \ldots, s_N)$ are local coordinates in U_α with

$$z_\alpha = f_{\alpha\beta}(z_\beta, s)$$

in $U_\alpha \cap U_\beta$ and

$$f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma, s), s) = f_{\alpha\gamma}(z_\gamma, s)$$

in $U_\alpha \cap U_\beta \cap U_\gamma$. So as not to have distracting indices floating around, we shall take the case $N = 1$ where s is a local coordinate on S; the extension to general N will be obvious. Setting

$\theta_{\alpha\beta} = \partial_s f_{\alpha\beta}$ the usual Kodaira-Spencer class associated to (2.8) is given by the composite map

$$\partial/\partial s \longrightarrow \left\{\theta_{\alpha\beta}\frac{\partial}{\partial z_\alpha}\right\} \in Z^1(\{U_\alpha\}, \Theta_{\mathfrak{X}/S}) \longrightarrow H^1(\Theta_{\mathfrak{X}/S}) \ .$$

With the notation $j_{\alpha\beta}^{-1} = \partial_{z_\beta} f_{\alpha\beta}$, the transition functions for $\Theta_{\mathfrak{X}}$ are

$$J_{\alpha\beta} =: \begin{pmatrix} j_{\alpha\beta}^{-1} & -j_{\alpha\beta}^{-1}\theta_{\alpha\beta} \\ 0 & 1 \end{pmatrix} \ .$$

This means that in $U_\alpha \cap U_\beta$

$$(\partial/\partial z_\alpha, \partial/\partial s) = (\partial/\partial z_\beta, \partial/\partial s)\, J_{\alpha\beta} \ .$$

Next, by *data* for the situation (2.1) we shall mean

$$\{U_\alpha, z_\alpha, f_{\alpha\beta}(z_\beta, s), \xi_{\alpha\beta}(z_\beta, s)\}$$

where $\{U_\alpha, z_\alpha, f_{\alpha\beta}(z_\beta, s)\}$ is data for $\mathfrak{X} \to S$ and $\xi_{\alpha\beta}(z_\beta, s)$ are transition functions for $\mathcal{L} \to \mathfrak{X}$. In terms of the data for (2.1) the transition functions for $\Sigma_{\mathcal{L}} \to \mathfrak{X}$ are

$$\text{(2.9)} \qquad \begin{pmatrix} 1 & \partial_{z_\beta} \log \xi_{\alpha\beta} & \partial_s \log \xi_{\alpha\beta} \\ 0 & j_{\alpha\beta}^{-1} & -j_{\alpha\beta}^{-1}\theta_{\alpha\beta} \\ 0 & 0 & 1 \end{pmatrix} \ .$$

The Kodaira-Spencer mapping

$$T \to H^1(\Sigma_L)$$

is represented in terms of the data associated to (2.1) by

$$\text{(2.10)} \qquad \partial/\partial s \to \{(\partial_s \log \xi_{\alpha\beta}, -\theta_{\alpha\beta}\partial/\partial z_\alpha)\} \ ,$$

where the term in braces is a Čech cocycle in $Z^1(\{U_\alpha\}, \Sigma_L)$. Here we are using the local splittings

$$\Sigma_L \mid_{U_\alpha} \cong (\mathcal{O}_X \mid_{U_\alpha}) \oplus (\Theta_X \mid_{U_\alpha})$$

given by the data.

In terms of the data $\{U_\alpha, z_\alpha, f_{\alpha\beta}(z_\beta, s), \xi_{\alpha\beta}(z_\beta, s)\}$ the Chern class $c_1(\mathcal{L})$ is represented by the Čech cocycle

$$\text{(2.11)} \qquad \{d \log \xi_{\alpha\beta}(z_\beta, s)\} \in Z^1(\{U_\alpha\}, \Omega^1_{\mathfrak{X}}) \ .$$

From (2.10) we see that the "horizontal" or ds component of (2.11) is the "$\mathcal{O}_X$-component" of the Kodaira-Spencer class associated to the family $L_s \to X_s$. Although this statement does not have intrinsic meaning, it is the "principal part" of the reason behind the result we are proving.

Since $\deg L_s = 0$ we may find $\{\varphi_\alpha\} \in \Omega^1_{\mathfrak{X}/S}(U_\alpha)$ such that in $U_\alpha \cap U_\beta$

$$d \log \xi_{\alpha\beta} \equiv \varphi_\alpha - \varphi_\beta \mod ds \ .$$

Over U_α we write sections of $\Omega^1_{\mathfrak{X}}$ as column vectors

$$\begin{pmatrix} \psi \\ \varphi \end{pmatrix}$$

where φ is the dz_α-component and ψ is the ds component, thus splitting the exact sequence

$$0 \to \pi^*\Omega^1_S \otimes \mathcal{O}_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}/S} \to 0$$

over U_α in terms of the given data. It follows from (2.9) that

$$\begin{aligned} \delta\left\{\begin{pmatrix} 0 \\ \varphi_\alpha \end{pmatrix}\right\} &= \begin{pmatrix} 0 \\ \varphi_\alpha \end{pmatrix} - \begin{pmatrix} 1 & \theta_{\alpha\beta}\partial/\partial z_\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \varphi_\beta \end{pmatrix} \\ &= \begin{pmatrix} (-\theta_{\alpha\beta}\partial/\partial z_\alpha \rfloor \varphi_\beta) ds \\ \varphi_\alpha - \varphi_\beta \end{pmatrix} \end{aligned}$$

where the brackets on the LHS refer to the value over $U_\alpha \cap U_\beta$ of the coboundary of the 0-cochain whose value over U_α is $\binom{0}{\varphi_\alpha}$. This will be our general notation for Čech computations. For the representative (2.11) of $c_1(\mathcal{L})$ we therefore have

$$\begin{pmatrix} \partial_s \log \xi_{\alpha\beta} \\ \partial_{z_\beta} \log \xi_{\alpha\beta} \end{pmatrix} - \delta\left\{\begin{pmatrix} 0 \\ \varphi_\alpha \end{pmatrix}\right\} = \begin{pmatrix} \partial_s \log \xi_{\alpha\beta} + (\theta_{\alpha\beta}\partial/\partial z_\alpha \rfloor \varphi_\beta) \\ 0 \end{pmatrix} .$$

It follows that

$$\{(\partial_s \log \xi_{\alpha\beta} + (\theta_{\alpha\beta}\partial/\partial z_\alpha)\rfloor\varphi_\beta)\, ds\} \in \Omega^1_S \otimes Z^1(\{U_\alpha\}, \mathcal{O}_X) \tag{2.12}$$

is a Čech representative of $c_1(\mathcal{L})$ in $F^1H^2(\mathfrak{X}, \mathbb{C}) = F^1\mathbb{H}^2(\Omega^\bullet_{\mathfrak{X}})$, where F^1 is the first step in the Leray filtration. From the definition in the Appendix, this expression for $c_1(\mathcal{L})$ gives a representative of $\delta\nu_L$.

On the other hand, evaluating at s_0 we recognize (2.12) as the image in $H^1(\mathcal{O}_X)$ of the Kodaira-Spencer class (2.10) under the decomposiion

$$H^1(\Sigma_L) \cong H^1(\mathcal{O}_X) \oplus H^1(\Theta_X) \tag{2.13}$$

induced by the splitting $\Sigma_L \cong \mathcal{O}_X \oplus \Theta_X$ given by writing the extension class $c_1(L)$ of (2.5) as the coboundary of $\{\varphi_\alpha\}$.

Since φ_α is unique up to adding an element of $H^0(\Omega^1_X)$, we see that the cocycle (2.12) gives a well-defined element

$$[\tau] \in T^* \otimes V^*/\nabla V \ .$$

By what was just said, this is the same as $\delta\nu_L$, which proves our theorem. □

Remark: This argument proves a little more. Denote by $\sigma \in T^* \otimes H^1(\Sigma_L)$ the Kodaira-Spencer class. Then in terms of the data

$$\{U_\alpha, z_\alpha, f_{\alpha\beta}(z_\beta, s), \xi_{\alpha\beta}(z_\beta, s), \varphi_\alpha(z_\alpha, s)\} \tag{2.14}$$

we have *canonical* representatives of $\tau \in T^* \otimes H^1(\mathcal{O}_X)$ and of the restriction along X of $c_1(\mathcal{L}) \in F^1\mathbb{H}^2(\Omega^\bullet_{\mathcal{X}})$, where we have written τ as the 1st component of σ using the decomposition (2.13) given by the data (2.14).

(c) We shall now give a geometric application of the preceding result. First some preliminaries that will be used in the proofs of Theorems 1 and 2.

For a point $p \in X$ we consider the exact sheaf sequence

$$0 \to \Theta_X \to \Theta_X(p) \to \Theta_X(p)\mid_p \to 0 \ . \tag{2.15}$$

Here, $\Theta_X(p)\mid_p$ is a skyscraper sheaf supported at p; upon choosing a local coordinate z centered at p we have

$$\Theta_X(p)\mid_p \cong \mathbb{C}$$

with $1/z\,\partial/\partial z$ representing a generator.

Definition: A *Schiffer* variation is class $\theta_p \in H^1(\Theta_X)$ given as the coboundary of $1/z\,\partial/\partial z$ in the cohomology sequence of (2.15).

We note that θ_p is well-defined up to scaling. We write

$$\theta_p = \delta\left\{\left(\frac{1}{z}\partial/\partial z\right)\right\} \tag{2.16}$$

and interpret this equation as follows: Relative to an open covering $U_0, U_1, \dots, U_m$ of X where $p \in U_0$, $p \notin U_1, \dots, U_m$, and z is a local coordinate in U_0 centered at p, there is an obvious 0-cochain given by $1/z\, \partial/\partial z$ in U_0 and zero elsewhere. Then (2.16) is the cohomology class associated to the cocycle in $Z^1(\{U_\alpha\}, \Theta_X)$ given by the coboundary of this cochain.

The map

$$H^0(\Omega^1_X) \xrightarrow{\theta_p} H^1(\mathcal{O}_X)$$

induced by the Gauss-Manin connection ∇ is easy to understand. In fact, we have

$$\ker \theta_p = H^0\left(\Omega^1_X(-p)\right) . \tag{2.17}$$

To verify this, if $\omega \in H^0(\Omega^1_X(-p))$ then using the obvious notation

$$\theta_p(\omega) = \delta\left\{\left(\frac{1}{z}\partial/\partial z \rfloor \omega\right)\right\}$$

and $(\frac{1}{z}\partial/\partial z\rfloor\omega)$ is holomorphic. To see the converse, we use duality. If $\varphi \in H^0(\Omega^1_X)$ then

$$\langle\theta_p(\omega), \varphi\rangle = \operatorname{Res}_p\left(\left(\frac{1}{z}\partial/\partial z\rfloor\omega\right)\varphi\right) .$$[2]

If this vanishes when $\varphi(p) \neq 0$, which we may arrange, it follows that $\omega(p) = 0$.

Geometrically, the image of the bicanonical map

$$\varphi_{2K} : X \to \mathbb{P}H^1(\Theta_X)$$

gives a curve in the projectivized tangent space to moduli, and the points on this curve are the Schiffer variations. Intuitively, they represent tangents to deformations of complex structure that leave $X - \{p\}$ unchanged and change the structure of X by a δ-function at p. In terms of VHS, for X non-hyperelliptic the Schiffer variations give the rank one transformations $H^0(\Omega^1_X) \xrightarrow{\theta} H^1(\mathcal{O}_X)$ where $\theta \in H^1(\Theta_X)$.

Next, we shall extend the discussion of Schiffer variations to pairs (X, L). For this we define

$$\Sigma_L(2p, p) \subset \Sigma_L(2p)$$

[2] The general principal is this: If $\xi \in H^1(\mathcal{O}_X)$ is written relative to the covering $\{U_\alpha\}$ as $\xi = \delta\eta$ where $\eta = \{\eta_\alpha\}$ may have poles, then for $\varphi \in H^0(\Omega^1_X)$

$$\langle\xi, \varphi\rangle = \Sigma \operatorname{Res}(\eta_\alpha\varphi).$$

The sum is over all the poles, and we note that for a pole at $p \in U_\alpha \cap U_\beta$ we have $\operatorname{Res}_p(\eta_\alpha\varphi) = \operatorname{Res}_p(\eta_\beta\varphi)$.

to be those $\sigma \in \Sigma_L(2p)$ that map to $\Theta_X(p)$. Intuitively, $\Sigma_L(2p,p)$ consists of those sections of Σ_L that have a 2^{nd}-order pole at p in the $\mathcal{O}_X$-component and a 1^{st}-order pole in the Θ_X-component. More precisely, the diagram

$$\begin{array}{ccccccccc} & & & & & & \Theta_X(p) & & \\ & & & & & & \downarrow & & \\ 0 & \to & \mathcal{O}_X(2p) & \to & \Sigma_L(2p) & \to & \Theta_X(2p) & \to & 0 \end{array}$$

pulls back to give a diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_X(2p) & \to & \Sigma_L(2p,p) & \to & \Theta_X(p) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_X(2p) & \to & \Sigma_L(2p) & \to & \Theta_X(2p) & \to & 0. \end{array}$$

In the diagram

$$\begin{array}{l} H^0(\Sigma_L(2p,p)\mid_p) \to H^0(\Sigma_L(2p)\mid_p) \to H^1(\Sigma_L) \\ \quad\;\; {\scriptstyle \pi}\downarrow \\ H^0(\Theta_X(p)\mid_p) \end{array}$$

we let $\sigma \in H^1(\Sigma_L)$ be the image of a $\zeta \in H^0(\Sigma_L(2p,p)\mid_p)$ which maps to a non-zero element of $H^0(\Theta_X(p)\mid_p)$ under π. Intuitively, using the above notation, if

$$\sigma = \delta\left\{\left(\frac{1}{z^2}, \frac{\lambda}{z}\frac{\partial}{\partial z}\right)\right\}$$

then

$$\sigma = (\sigma', \sigma'')$$

where $\sigma'' \in H^1(\Theta_X)$ is a Schiffer variation and

$$\sigma' \in \text{Image}\left\{H^0(\mathcal{O}_X(2p)\mid_p) \to H^1(\mathcal{O}_X)\right\}$$

is non-zero. If $\omega \in H^0(\Omega^1_X(-p))$, then

$$\sigma \otimes \omega \in \ker\{T \otimes V \to V^*\}$$

by the above discussion. By the theorem we have

$$\begin{aligned} \delta\nu_L(\sigma \otimes \omega) &= \langle \sigma', \omega \rangle \\ &= \text{Res}_p\left(\frac{1}{z^2}\omega\right) \\ &=: \omega'(p)\ . \end{aligned}$$

Since, if $g \geq 2$ we may choose ω with $\omega'(p) \neq 0$ this gives the

Corollary: *For any situation (2.1) where $g \geq 2$ and for some $s \in S$ we have that the image of*

$$T_s S \to H^1(\Sigma_{L_s})$$

contains a Schiffer variation coming from $\Sigma_L(2p, p) \mid_p$ as above, it follows that

$$\delta \nu_{\mathcal{L}} \neq 0 \ .$$

For us this result is interesting in that the method of proof provides a "toy model" for the argument to be given in Theorem 2 below.

Remark: In terms of the Čech representative (2.12) expressed in terms of the data for (2.1) together with what might be called the *auxilary data* $\{\varphi_\alpha\}$ where

$$\partial_{z_\beta} \log \xi_{\alpha\beta} = \delta\{\varphi_\alpha\} \ ,$$

if we write

$$\left(\partial_s \log \xi_{\alpha\beta} + \left(\theta_{\alpha\beta} \frac{\partial}{\partial z_\alpha}\right) \rfloor \varphi_\beta, \theta_{\alpha\beta} \frac{\partial}{\partial z_\alpha}\right) = \delta \left\{ \left(\frac{1}{z^2}, \frac{1}{z} \frac{\partial}{\partial z} \right) \right\}$$

then

$$\{\partial_s \log \xi_{\alpha\beta}\} \equiv \delta \left\{ \frac{1}{z^2} \right\} \ \text{modulo} \ \frac{1}{z}\text{-terms}$$

defines a 1^{st}-order s-variation of the transition functions $\xi_{\alpha\beta}(z_\beta)$. We note that the

$$\langle \sigma', \omega \rangle$$

depends only on the data for (2.1), and not on the auxiliary data. This is because $\{\varphi_\alpha\}$ is unique up to adding $\psi \in H^0(\Omega^1_X)$, and then by (2.12) $\langle \sigma', \omega \rangle$ changes by adding

$$\begin{aligned} \left\langle \delta \left\{ \left(\frac{1}{z} \frac{\partial}{\partial z} \right) \rfloor \psi \right\}, \omega \right\rangle &= \operatorname{Res}_p \left(\frac{1}{z} \omega \right) \\ &= \omega(p) \\ &= 0 \ . \end{aligned}$$

The reason this is relevant is the following: For the canonical bundle K_X there is a canonical $\mathbb{C}$-*linear* lift

$$\Theta_X \xrightarrow{j} \Sigma_{K_X}$$

given by the Lie derivative

$$j(v)\varphi = \mathcal{L}_v \varphi$$

where $v \in \Theta_X$ and $\varphi \in \mathcal{O}(K_X) = \Omega^1_X$. This gives a canonical lifting

$$H^1(\Theta_X) \xrightarrow{j} H^1(\Sigma_{K_X}) \ .$$

Suppose now that $\theta_p = \delta\left\{\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\right\}$ is a Schiffer variation. From the formula

$$\begin{aligned}
\mathcal{L}_{\frac{1}{z}\frac{\partial}{\partial z}}(f(z)dz) &= d\left(\frac{f(z)}{z}\right) \\
&= \left(-\frac{1}{z^2}\right) f(z)dz + \frac{f'(z)}{z}dz
\end{aligned}$$

we see that

$$j(\theta_p) = \delta\left\{\left(-\frac{1}{z^2}, \frac{1}{z}\frac{\partial}{\partial z}\right)\right\} \ .$$

Thus, *the principal part of the $\mathcal{O}_X$-component of $j(\theta_p)$ is $\delta\left(-\frac{1}{z^2}\right)$*. Although this statement does not have intrinsic meaning, it provides an heuristic that is central to our main calculation.

3 Proof of Theorem 1

In this section we will give the proof of Theorem 1. Again, we will proceed in two steps, giving first a calculation in the curve case and then extending that calculation to the surface case. These computations, which were motivated by [9], may be of interest in their own right.

(a) We assume given the situation

$$\begin{array}{c} \mathcal{D} \subset \mathfrak{X} \\ \qquad\downarrow \\ \qquad S \end{array} \tag{3.1}$$

where $\mathfrak{X} \to S$ is a family $\{X_s\}_{s\in S}$ of smooth curves and $\mathcal{D} = \{D_s = \sum_\lambda n_\lambda p_\lambda(s) : \sum_\lambda n_\lambda = 0\}$ is a family of divisors of degree zero. Here, $s \to p_\lambda(s)$ is a section of (3.1). Given the situation (2.1) we may have to pass to finite covering to be able to define divisors D_s with $[D_s] = L_s$. In the preceding section we gave a method for calculating the infinitesimal invariant $\delta\nu_{\mathcal{D}}$ in terms of the variational data associated to the family of line bundles $[D_s] \to X_s$. In this section we shall give an alternate method for calculating $\delta\nu_{\mathcal{D}}$ geometrically in terms of the divisors D_s.

Setting $D = D_{s_0}$ and recalling our notations

$$\begin{cases} T = T_{s_0} S \\ V = H^0(\Omega^1_X) \\ V^* = H^1(\mathcal{O}_X) \end{cases}$$

we have for value $\delta\nu_D$ of $\delta\nu_{\mathcal{D}}$ at $s = s_0$ that

$$\delta\nu_D \in \left\{ \ker\{T \otimes V \to V^*\} \right\}^* .$$

If $\dim S = N$, then using

$$T \otimes V \cong \Lambda^N T \otimes \Lambda^{N-1} T^* \otimes V$$

we may think of $\gamma \in T \otimes V$ as a section along X of the bundle

$$K_S^{-1} \otimes \left(\Omega_S^{N-1} \otimes \Omega^1_{\mathfrak{X}/S} \right) .$$

In terms of coordinates as in the preceding section, if $\gamma = \sum_i \partial/\partial s_i \otimes \omega_i$ where $\omega_i \in H^0(\Omega^1_X)$ is given by $\{g_{i\alpha}(z_\alpha) dz_\alpha\}$, then in U_α

$$\gamma = (\partial/\partial s_1 \otimes \cdots \otimes \partial/\partial s_N) \otimes \left(\sum_i ds_{(i)} \otimes g_{i\alpha}(z_\alpha) dz_\alpha \right)$$

where $ds_{(i)} = (-1)^{i-1} ds_1 \wedge \cdots \wedge \widehat{ds_i} \wedge \cdots \wedge ds_N$. Now

$$\Omega_S^{N-1} \otimes \Omega^1_{\mathfrak{X}/S} \cong \Omega^N_{\mathfrak{X}} / \Omega^N_S . \tag{3.2}$$

Lemma: $\gamma \in \ker\{T \otimes V \to V^*\}$ *is the value at* s_0 *of the image under (3.2) of a section*

$$(\partial/\partial s_1 \wedge \cdots \wedge \partial/\partial s_N) \otimes \Gamma$$

where $\Gamma \in H^0(\Omega^N_{\mathfrak{X}})$. *Moreover,* Γ *is unique up to adding a section in* $H^0(\mathfrak{X}, \Omega^N_S)$.

Here we allow ourselves to shrink S.

Assuming the lemma we then have the

Proposition 1: *The value* $\delta\nu_D(\gamma)$ *is given by*

$$\delta\nu_D(\gamma) = (\partial/\partial s_1 \wedge \cdots \wedge \partial/\partial s_N) \rfloor \left(\sum_\lambda n_\lambda p_\lambda^*(\Gamma) \right) . \tag{3.3}$$

Here, $p_\lambda^*(\Gamma)$ is the pullback of Γ under the section

$$s \to p_\lambda(s) \in X_s$$

of $\mathcal{X} \to S$, and it is understood that the RHS of (3.3) is evaluated at s_0. We note that since $\sum_\lambda n_\lambda = 0$, the RHS of (3.3) is unchanged if we add to Γ a section of Ω_S^N.

Before presenting the proofs of the lemma and proposition we shall give the

Example: We assume that X is a fixed curve and D_s is a family of divisors parametrized by $S = \mathrm{Pic}^0(X)$ with $[D_s] = s$. Then $T = H^1(\mathcal{O}_X) = V^*$ and the map $T \otimes V \to V^*$ is trivial. For

$$\gamma = (\partial/\partial s_1 \wedge \cdots \wedge \partial/\partial s_N) \otimes ds_{(i)} \otimes \omega$$

where $\omega \in H^0(\Omega_X^1)$, we may take

$$\Gamma = ds_{(i)} \wedge \omega$$

and then

$$p_\lambda^*(\Gamma) = \left\langle \omega, \frac{\partial p_\lambda}{\partial s_i} \right\rangle ds_1 \wedge \cdots \wedge ds_N$$

where $\frac{\partial p_\lambda}{\partial s_i} \in T_{p_\lambda} X$ is the evident tangent vector. It follows from the proposition that

$$\delta \nu_D \in V \otimes V^*$$

is the identity, thereby establishing the assertion made in the example just below $(2.3)^*$ in the preceding section.

Proof of the lemma: We have on $\mathcal{X}$ the exact sheaf sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \to K_S^{-1} \otimes \Omega_{\mathcal{X}}^N \to K_S^{-1} \otimes \Omega_S^{N-1} \otimes \Omega_{\mathcal{X}/S}^1 \to 0 \,. \tag{3.4}$$

Recalling that we are working in the neighborhood of a generic point of S where all direct image sheaves are locally free and maps between them have constant rank, the exact cohomology sequence of the direct images of (3.4) gives a complex of vector bundles whose value at s_0 is

$$\begin{array}{ccccc} 0 \to \mathbb{C} \to \Lambda^N T \otimes H^0\left(\Omega_{\mathcal{X}}^N \otimes \mathcal{O}_X\right) \to & \underbrace{\Lambda^N T \otimes \Lambda^{N-1} T^* \otimes H^0(\Omega_X^1)} & \to & H^1(\mathcal{O}_X) \\ & \| & & \| \\ & T \otimes V & \longrightarrow & V^*. \end{array}$$

It follows that $\gamma \in \ker\{T \otimes V \to V^*\}$ lifts to $\Lambda^N T \otimes H^0\left(\Omega_{\mathcal{X}}^N \otimes \mathcal{O}_X\right)$, and this is then the restriction along X of a section in $H^0(\mathcal{X}, K_S^{-1} \otimes \Omega_{\mathcal{X}}^N)$. $\square$

Proof of Proposition 1: Arguing as in the proof of (A.9) in the Appendix, we may choose data for (3.1) where $[\mathcal{D}] = \mathcal{L}$, $\deg L_s = 0$ to be

$$\{U_\alpha, z_\alpha, f_{\alpha\beta}(z_\beta, s), \xi_{\alpha\beta}(s)\} ,$$

where the transition functions for $L_s \to X_s$ are constant along X_s. Our assumption that $L_s = [D_s]$ where

$$D_s = \sum_\lambda n_\lambda p_\lambda(s)$$

with

$$s \to p_\lambda(s)$$

being a holomorphic section of $\mathfrak{X} \to S$ means that there are in U_α meromorphic functions $\zeta_{\alpha\lambda}(z_\alpha, s)$ such that

$$\text{(3.5)} \qquad \begin{cases} \xi_{\alpha\beta}(s) = \prod_\lambda \zeta_{\alpha\lambda}(f_{\alpha\beta}(z_\beta, s), s)^{n_\lambda} \prod_\mu \zeta_{\beta\mu}(z_\beta, s)^{-n_\mu} \\ \zeta_{\alpha\lambda}(p_{\alpha\lambda}(s), s) = 0 \end{cases}$$

where $p_\lambda(s)$ is given in U_α by $z_\alpha = p_{\alpha\lambda}(s)$ with

$$(\partial_{z_\alpha} \zeta_{\alpha\lambda})\,(p_{\alpha\lambda}(s), s) \neq 0 .$$

Now from the Appendix we have that $\delta\nu_{\mathcal{L}} \in \Omega^1_S \otimes R^1_\pi \mathcal{O}_{\mathfrak{X}} / \nabla R^0_\pi \Omega^1_{\mathfrak{X}/S}$ is represented by the Čech cocycle

$$\{d \log \xi_{\alpha\beta}(s)\} .$$

Setting

$$\zeta_\alpha = \prod_\lambda \zeta_{\alpha\lambda}^{n_\lambda}$$

we have

$$\text{(3.6)} \qquad d \log \xi_{\alpha\beta}(s) = \delta\left\{\partial_s \log \zeta_\alpha\right\} .$$

Evaluating at s_0, we have written a Čech representative of $\delta\nu_L \in T^* \otimes H^1(\mathcal{O}_X)/\nabla H^0(\Omega^1_X)$ as the coboundary of a cochain with poles.

Now an element

$$\gamma \in \ker\{T \otimes V \to V^*\}$$

is given as

$$\gamma = \sum_i ds_i \otimes \omega_i$$

where $\omega_i = \{\omega_{i\alpha} = g_{i\alpha}(z_\alpha)dz_\alpha\} \in H^0(\Omega^1_X)$. Then

$$\delta\nu_L(\gamma) = \sum_i \langle \partial_{s_i} \log \xi_{\alpha\beta}(s), \omega_i \rangle$$

where the RHS is the duality pairing

$$Z^1(\{U_\alpha\}, \mathcal{O}_X) \otimes H^0(\Omega^1_X) \to \mathbb{C} .$$

By (3.5) this gives

$$\delta\nu_L(\gamma) = \sum_i \operatorname{Res}(\partial_{s_i} \log \zeta_\alpha \cdot \omega_{i\alpha}) .$$

On the other hand, by (3.5)

$$\partial_{s_i} \log \zeta_\alpha + \sum_\lambda n_\lambda (\partial_{z_\alpha} \log \zeta_{\alpha\lambda} \partial_{s_i} p_{\alpha\lambda}) = 0 .$$

Thus

$$(3.7) \qquad \begin{aligned} \sum_i \operatorname{Res}_{p_{\alpha\lambda}} (\partial_{s_i} \log \zeta_\alpha \omega_i) &= -\sum_i \operatorname{Res}_{p_{\alpha\lambda}} n_\lambda (\partial_{z_\alpha} \log \zeta_{\alpha\lambda} \partial_{s_i} p_{\alpha\lambda} \omega_{i\alpha}) \\ &= -\sum_i n_\lambda \underbrace{(\partial_{s_i} p_{\alpha\lambda}(s) g_{i\alpha}(p_{\alpha\lambda}(s)))} \end{aligned}$$

where the term over the braces is to be evaluated at $s = s_0$ and $z_\alpha = p_{\alpha\lambda}(s_0)$.

Now we may assume that

$$\gamma = (\partial/\partial s_1 \wedge \cdots \wedge \partial/\partial s_N) \otimes \sum ds_{(i)} \otimes \omega_i$$

comes from a section $\partial/\partial s_1 \wedge \cdots \wedge \partial/\partial s_N \otimes \Gamma$ of $K_S^{-1} \otimes H^0(\Omega^1_{\mathfrak{X}} \otimes \mathcal{O}_X)$ where in U_α

$$\Gamma \equiv \sum_i ds_{(i)} \wedge \omega_{i\alpha} \text{ modulo } ds_1 \wedge \cdots \wedge ds_N .$$

Then

$$p^*_{\alpha\lambda}(\Gamma) = \left(\sum_i \partial_{s_i} p_{\alpha\lambda}(s_0) g_{i\alpha}(p_{\alpha\lambda}(s_0)) \right) ds_1 \wedge \cdots \wedge ds_N ,$$

and comparing with (3.7) gives our result. □

(b) We now want to extend the preceding result to the situation

$$(3.8) \qquad \begin{array}{c} \mathcal{Z} \subset \mathcal{Y} \\ \downarrow \\ S \end{array}$$

of a family of 0-cycles $\{z_s\}_{s\in S}$ on a family $\{Y_s\}_{s\in S}$ of smooth surfaces with

$$\left\{ \begin{array}{l} \deg z_s = 0 \\ \mathrm{Alb}_{Y_s}(z_s) = 0 \; . \end{array} \right. \tag{3.9}$$

The formulation is analogous to what was just done. Letting s_0 be a generic point and $z = z_{s_0}$, $Y = Y_{s_0}$ with the notations

$$\left\{ \begin{array}{l} T = T_{s_0} S \\ W = H^0(\Omega^2_Y) \\ U = H^1(\Omega^1_Y) \end{array} \right.$$

we have for the value $\delta\nu_z$ of $\delta\nu_{\mathcal{Z}}$ at s_0 that

$$\delta\nu_z \in \ker \left\{ \Lambda^2 T \otimes W \xrightarrow{\nabla} T \otimes U \right\} \; .$$

By linear algebra, if $\dim T = N$ then

$$\Lambda^2 T \cong \Lambda^N T \otimes \Lambda^{N-2} T^* \tag{3.10}$$

and we may think of $\xi \in \Lambda^2 T \otimes W$ as a section along Y of the bundle

$$K_S^{-1} \otimes \left(\Omega_S^{N-2} \otimes \Omega^2_{\mathcal{Y}/S} \right) \; .$$

Lemma: *Under the assumption (3.9), $\gamma \in \ker\{\Lambda^2 T \otimes W \to T \otimes U\}$ is the value at s_0 of the image of a section*

$$(\partial/\partial s_1 \wedge \cdots \wedge \partial/\partial s_N) \otimes \Gamma$$

where $\Gamma \in H^0(\Omega^N_{\mathcal{Y}})$. Moreover, $\Gamma|_Y$ is unique modulo the images of

(i) $\Lambda^N T^* \to H^0(\Omega^N_{\mathcal{Y}} \otimes \mathcal{O}_Y)$

(ii) $\Lambda^{N-1} T^* \otimes H^0(\Omega^1_Y) \to H^0 \left(\Omega^N_{\mathcal{Y}} \otimes \mathcal{O}_Y \right) / \Lambda^N T^*$.

The proof of this lemma is an extension of the argument used for the analogous result in the preceding section — it involves making the linear algebra identification $T \cong \Lambda^N T \otimes \Lambda^{N-1} T^*$ in (3.10) and working through the arguments leading to the construction of $\delta\nu_z$ in the appendix. We remark that the ambiguities in Γ will work out, (i) because of the assumption $\deg z = 0$ and (ii) because of the assumption $\mathrm{Alb}_Y(z) = 0$ (the ambiguity in (ii) is essentially the infinitesimal invariant associated to the normal function $s \to \mathrm{Alb}_{Y_s}(z_s)$).

We write

$$z_s = \sum_\lambda n_\lambda p_\lambda(s)$$

where $s \to p_\lambda(s)$ is a section of $\mathcal{Y} \to S$.

Proposition 2: *The value $\delta\nu_z(\gamma)$ is given by*

$$\delta\nu_z(\gamma) = (\partial/\partial s_1 \wedge \cdots \wedge \partial/\partial s_N)\rfloor \left(\sum_\lambda n_\lambda p_\lambda^*(\Gamma)\right) .$$

Assuming the proposition we will give the

Proof of Theorem 1: We will use the opportunity to establish a more general setting and then specialize to a situation that will give a proof of Theorem 1. Let $S = \mathrm{Pic}^n(\mathcal{X}/\mathcal{M}_g)$ be the set of line bundles of degree n over a curve of genus g (passing to finite branched coverings, etc). We then have the universal line bundle

$$\begin{array}{ccc} \mathcal{L} \to & \mathcal{X} \\ & \downarrow \\ & S . \end{array}$$

Let $\mathcal{Y} = \mathcal{X} \times_S \mathcal{X}$ and let

$$\begin{array}{ccc} \mathcal{Z} \subset & \mathcal{Y} \\ & \downarrow \\ & S \end{array}$$

a family of 0-cycles z_s representing the rational equivalence classes of the z_{L_s}'s associated to $L_s \to X_s$ on $Y_s = X_s \times X_s$ by the construction in the introduction; i.e., if $L_s = [D_s]$ then

$$z_s = D_s \times D_s - n\, D_{s,\Delta} .$$

It is clear that $\deg z_s = 0$, and we will verify that $\mathrm{Alb}_{Y_s}(z_s) = 0$. Dropping reference to the particular point s we write

$$D = \sum_\lambda n_\lambda p_\lambda .$$

Then

$$z = \sum_{\lambda,\mu} n_\lambda n_\mu (p_\lambda \times p_\mu) - n\left(\sum_\lambda n_\lambda p_\lambda \times p_\lambda\right)$$

where $n = \sum_\lambda n_\lambda$. Then

$$\mathrm{Alb}(Y) \cong J(X) \oplus J(X)$$

and

$$\begin{aligned} \mathrm{Alb}_Y(z) \;\; &= \;\; \left(\sum_{\lambda,\mu} n_\lambda n_\mu \mathrm{AJ}_X(p_\lambda), \sum_{\lambda,\mu} n_\lambda n_\mu \mathrm{AJ}_X(p_\mu)\right) \\ &\quad -n\left(\sum_\lambda n_\lambda\, \mathrm{Alb}_Y(p_\lambda \times p_\lambda)\right) . \end{aligned}$$

The last term is equal to

$$n\left(\sum_{\lambda} n_{\lambda}\mathrm{AJ}_X(p_{\lambda}), \sum_{\mu} n_{\mu}\mathrm{AJ}_X(p_{\mu})\right)$$

and $\mathrm{Alb}_{Y_s}(z_s) = 0$ follows.

We may show that $z_L \neq 0$ in $CH^2(Y)$ for general X and L by showing that the infinitesimal invariant

$$\delta\nu_{\mathcal{Z}} \neq 0\ . \tag{3.11}$$

We will now use the above proposition to prove (3.11), and will observe that the argument also gives a proof of Theorem 1 in that in the argument we can fix X and let L vary.

Evaluation of $\delta\nu_{\mathcal{Z}}$ at $L \to X$ gives

$$\delta\nu_z \in \Lambda^2 T^* \otimes H^2(\mathcal{O}_Y)/\nabla\left(T^* \otimes H^1(\Omega^1_Y)\right)$$

where $T = T_{s_0}S$. (The notation means that $\delta\nu_{\mathcal{Z}}(s_0) = \delta\nu_z$.) By duality

$$\delta\nu_z \in \left\{\ker\left\{\Lambda^2 T \otimes H^0(\Omega^2_Y) \to T \otimes H^1(\Omega^1_Y)\right\}\right\}^*\ .$$

Since the rational equivalence class of $\mathcal{Z}$ is invariant under the obvious involution $(p,q) \to (q,p)$ $(p,q \in X)$, it follows that $\delta\nu_z$ actually lies in the skew-symmetric part $\Lambda^2 H^1(\mathcal{O}_X)$ of

$$H^2(\mathcal{O}_Y) \cong H^1(\mathcal{O}_X) \otimes H^1(\mathcal{O}_X)\ .$$

Thus, setting $V = H^0(\Omega^1_X)$

$$\delta\nu_z \in \ker\left\{\Lambda^2 T \otimes \Lambda^2 V \to T \otimes U\right\}^*\ .$$

In order to prove (3.11) it will suffice to show that the restriction of $\delta\nu_{\mathcal{Z}}$ to a subfamily is non-zero. For our subfamily we fix X, and changing notation we now have $S = \mathrm{Pic}^n(X)$ and, since ∇ is zero on this subfamily and $T = H^1(\mathcal{O}_X) = V^*$

$$\delta\nu_z \in \Lambda^2 V^* \otimes \Lambda^2 V\ . \tag{3.12}$$

We will show that:

In (3.12), $\delta\nu_z$ is the identity.

Of course, if $\dim V = g = 1$ the statement is trivial. Only if $g \geqq 2$ do we have $\delta\nu_z \neq 0$. The case $g = 1$ will be discussed at the end of this section.

Proof: Write $D_s = \sum_{\lambda=1}^{n} p_\lambda(s)$, so that

$$z_s = \sum_{\lambda,\mu} p_\lambda(s) \times p_\mu(s) - n \sum_{\lambda} p_\lambda(s) \times p_\lambda(s) \ .$$

We will use the proposition to evaluate

$$\delta\nu_z \left(\partial/\partial s_i \wedge \partial/\partial s_j \otimes \omega \wedge \varphi\right) \ . \tag{3.13}$$

If X is given by data $\{U_\alpha, z_\alpha, f_{\alpha\beta}(z_\beta)\}$ then $Y = X \times X$ will be covered by open sets $U_\alpha \times U_\beta$ with product coordinates (z_α, w_β). Let ω be given by $\{g_\alpha(z_\alpha)dz_\alpha\}$ and φ by $\{h_\alpha(z_\alpha)dz_\alpha\}$. Then $\omega \wedge \varphi$ is given in $U_\alpha \times U_\beta$ by

$$\left(g_\alpha(z_\alpha)h_\beta(w_\beta) - h_\alpha(z_\alpha)g_\beta(w_\beta)\right) dz_\alpha \wedge dw_\beta =: k_{\alpha\beta}(z_\alpha, w_\beta)dz_\alpha \wedge dw_\beta \ . \tag{3.14}$$

Letting

$$s \to (z_\alpha(s), w_\beta(s)) \tag{3.15}$$

be one of the cross-sections $p_\lambda(s) \times p_\mu(s)$, the pullback of (3.14) under (3.15) is

$$\sum_{i,j} k_{\alpha\beta}(z_\alpha(s), w_\beta(s)) \left(\partial_{s_i} z_\alpha(s) \partial_{s_j} w_\beta(s) - \partial_{s_j} z_\alpha(s) \partial_{s_i} w_\beta(s)\right) ds_i \wedge ds_j \ . \tag{3.16}$$

When $\lambda = \mu$, this gives zero. It follows from this together with (3.16) and the proposition that

$$\delta\nu_z \left(\partial/\partial s_i \wedge \partial/\partial s_j \otimes \omega \wedge \varphi\right) = \left(\partial/\partial s_1 \wedge \cdots \wedge \partial/\partial s_N\right) \rfloor \sum_{\lambda \neq \mu} \left(p_\lambda(s) \times p_\mu(s)\right)^* ds_{(ij)} \wedge \omega \wedge \varphi$$

where $ds_{(ij)} = (-1)^{i+j} ds_1 \wedge \cdots \wedge \widehat{ds_i} \wedge \cdots \wedge \widehat{ds_j} \wedge \cdots \wedge ds_N$

$$= \sum_{\lambda,\mu} \left\langle \omega, \frac{\partial p_\lambda}{\partial s_i} \right\rangle \left\langle \varphi, \frac{\partial p_\mu}{\partial s_j} \right\rangle - \left\langle \omega, \frac{\partial p_\mu}{\partial s_j} \right\rangle \left\langle \varphi, \frac{\partial p_\lambda}{\partial s_i} \right\rangle \ . \tag{3.17}$$

But $\partial/\partial s_i \in T = H^1(\mathcal{O}_X) = H^0(\Omega_X^1)^*$ is given by

$$\omega \to \sum_{\lambda} \left\langle \omega, \frac{\partial p_\lambda}{\partial s_i} \right\rangle \ .$$

From (3.17) it follows then that

$$\delta\nu_z \in \Lambda^2 V \otimes \Lambda^2 V^*$$

is the "identity", as claimed. □

Proof of Proposition 2: Using an extension of the method to establish 1.14 in [9] we will reduce the result to the proposition given in Section 3(a) above.

Given the situation (3.8), by shrinking S and passing to finite coverings we may find a family of smooth ample curves $X_s \subset Y_s$ such that the 0-cycle z_s is supported on X_s. Thus we have a situation

$$\begin{array}{ccccc} \mathcal{Z} & \subset & \mathcal{X} & \subset & \mathcal{Y} \\ & & \downarrow \pi_{\mathcal{X}} & & \downarrow \pi_{\mathcal{Y}} \\ & & S & = & S\,. \end{array} \tag{3.18}$$

We will denote by

$$\delta\nu_{\mathcal{Z},\mathcal{X}} \in R^1_{\pi_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}} \otimes \Omega^1_S / \nabla_{\mathcal{X}}\left(R^0_{\pi_{\mathcal{X}}}\Omega^1_{\mathcal{X}/S}\right)$$

and

$$\delta\nu_{\mathcal{Z},\mathcal{Y}} \in R^2_{\pi_{\mathcal{Y}}}\mathcal{O}_{\mathcal{Y}} \otimes \Omega^2_S / \nabla_{\mathcal{Y}}\left(R^1_{\pi_{\mathcal{Y}}}\Omega^1_{\mathcal{Y}/S} \otimes \Omega^1_S\right)$$

the infinitesimal invariants associated to the families of 0-cycles $z_s \in Z^1(X_s)$ and $z_s \in Z^2(Y_s)$ respectively. We will show that:

$\delta\nu_{\mathcal{X},\mathcal{Y}}$ may be computed, in a natural way, from $\delta\nu_{\mathcal{Z},\mathcal{X}}$.

When Y is regular the result is proved in [9]. The main ingredients needed to extend her argument are the following:

(i) There is a surjection

$$J(X_s) \to \mathrm{Alb}(Y_s)\,.$$

In fact, there is an isogeny

$$J(X_s) \cong J_{\mathrm{var}}(X_s) \oplus \mathrm{Alb}(Y_s)$$

where $J_{\mathrm{var}}(X_s)$ is the "variable part" of $J(X_s)$ when X_s varies in Y_s, holding s fixed. On the tangent space level we have

$$H^1(\mathcal{O}_{X_s}) \cong H^1_{\mathrm{var}}(\mathcal{O}_{X_s}) \oplus H^1(\mathcal{O}_{Y_s})\,. \tag{3.19}$$

This is the projection onto Hodge $(0,1)$ components of the orthogonal (under the natural inner product) direct sum decomposition

$$H^1(X_s,\mathbb{C}) \cong H^1(Y_s,\mathbb{C}) \oplus H^1_{\mathrm{var}}(X_s,\mathbb{C}) \tag{3.20}$$

where by definition

$$H^1_{\mathrm{var}}(X_s,\mathbb{C}) = \mathrm{im}\left\{H^1(Y_s,\mathbb{C}) \to H^1(X_s,\mathbb{C})\right\}^{\perp}\,.$$

(ii) We now let $s \in S$ vary and consider the variations of Hodge structure associated to (3.18). Then the VHS associated to $\{H^1(Y_s, \mathbb{C}\}_{s\in S}$ is a sub-VHS of that associated to $\{H^1(X_s, \mathbb{C})\}_{s\in S}$, and by the *semi-simplicity of monodromy* the sub-VHS has a natural complement given by (3.20) that is invariant under monodromy. Thus we have

$$\begin{aligned} R^1_{\pi_{\mathfrak{X}}}\mathbb{C} &= R^1_{\pi_{\mathfrak{Y}}}\mathbb{C} \oplus R^1_{\pi_{\mathfrak{X}},\mathrm{var}}\mathbb{C} \\ R^1_{\pi_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{X}} &= R^1_{\pi_{\mathfrak{Y}}}\mathcal{O}_{\mathfrak{Y}} \oplus R^1_{\pi_{\mathfrak{X}},\mathrm{var}}\mathcal{O}_{\mathfrak{X}}\ . \end{aligned} \tag{3.21}$$

To formulate the result we work at a generic point s_0 of S and use our earlier notations $X = X_{s_0}$, $Y = Y_{s_0}$, $T = T_{s_0}S$. Setting

$$U = Y \setminus X$$

there is a standard exact sequence ($\mathbb{C}$-coefficients)

$$0 \to H^0(X) \to H^2(Y) \to H^2(U) \xrightarrow{\mathrm{Res}} H^1(X) \to H^3(Y) \to 0 \tag{3.22}$$

and

$$\mathrm{Res}\, H^2(U) = H^1_{\mathrm{var}}(X)\ .$$

Now (3.22) is an exact sequence of mixed Hodge structures with

$$\mathrm{Res}\, F^i H^2(U) = F^{i-1} H^1_{\mathrm{var}}(X)$$

and

$$\begin{aligned} F^1H^2(U)/F^2H^2(U) &\cong H^1\left(\Omega^1_Y(\log X)\right) \\ F^0H^2(U)/F^1H^2(U) &\cong H^2(\mathcal{O}_Y)\ . \end{aligned}$$

From the fact that when s varies the whole sequence (3.22) varies to give VMHS's, we have

$$H^1\left(\Omega^1_Y(\log X)\right) \xrightarrow{\nabla_U} H^2(\mathcal{O}_Y)\otimes T^*\ .$$

By linear algebra this leads to a diagram

$$\begin{array}{ccc} H^1(\Omega^1_Y)\otimes T^* & \xrightarrow{\nabla_Y} & H^2(\mathcal{O}_Y)\otimes \Lambda^2 T^* \\ \downarrow & & \| \\ H^1\left(\Omega^1_Y(\log X)\right)\otimes T^* & \xrightarrow{\nabla_U} & H^2(\mathcal{O}_Y)\otimes \Lambda^2 T^* \\ \downarrow \mathrm{Res} & & \\ H^1_{\mathrm{var}}(\mathcal{O}_X)\otimes T^* & & \\ \downarrow & & \\ 0\ . & & \end{array} \tag{3.23}$$

On the other hand we have

$$
(3.24)\qquad
\begin{array}{ccc}
H^0\left(\Omega^2_Y(\log X)\right) & \xrightarrow{\nabla_U} & H^1\left(\Omega^1_Y(\log X)\right)\otimes T^* \\
\downarrow \mathrm{Res} & & \downarrow \mathrm{Res} \\
H^0_{\mathrm{var}}\left(\Omega^1_X\right) & \xrightarrow{\nabla_X} & H^1_{\mathrm{var}}(\mathcal{O}_X)\otimes T^* \\
\downarrow & & \\
0\ . & &
\end{array}
$$

Using $\nabla_U \cdot \nabla_U = 0$, again by linear algebra the diagrams (3.23) and (3.24) lead to a map, induced by ∇_U in (3.23),

$$
(3.25)\qquad \frac{H^1_{\mathrm{var}}(\mathcal{O}_X)\otimes T^*}{\nabla_X H^0_{\mathrm{var}}(\Omega^1_X)} \xrightarrow{\nabla_U} \frac{H^2(\mathcal{O}_Y)\otimes \Lambda^2 T^*}{\nabla_Y\left(H^1(\Omega^1_Y)\otimes T^*\right)}\ .
$$

Lemma: *$\delta\nu_{z,X} \in H^1_{\mathrm{var}}(O_X)\otimes T^*/\nabla_X(H^0_{\mathrm{var}}(\Omega^1_X))$ and*

$$\nabla_U(\delta\nu_{z,X}) = \delta\nu_{z,Y}$$

in (3.25).

Proof: The assumption that

$$\mathrm{Alb}_{Y_s}(z_s) = 0$$

means that

$$\nu_{\mathcal{Z},\mathcal{X}}(s) \in \ker\{J(X_s) \to \mathrm{Alb}(Y_s)\}\ .$$

Using the discussion above together with the fact (3.19) that the tangent spaces naturally split and that the construction of $\delta\nu_{\mathcal{Z},\mathcal{X}}$ only "sees" tangent spaces we infer that

$$\delta\nu_{z,X} \in H^1_{\mathrm{var}}(\mathcal{O}_X)\otimes T^*/\nabla_X\left(H^0_{\mathrm{var}}(\Omega^1_X)\right)\ .$$

From the exact cohomology sequence of

$$(3.26)\qquad 0 \to \Omega^\bullet_{\mathcal{Y}} \to \Omega^\bullet_{\mathcal{Y}}(\log \mathcal{X}) \xrightarrow{\mathrm{Res}} \Omega^{\bullet-1}_{\mathcal{X}} \to 0$$

we infer a map

$$(3.27)\qquad \mathbb{H}^1\left(\Omega^{\geqq 1}_{\mathcal{X}}\right) \to \mathbb{H}^2\left(\Omega^{\geqq 2}_{\mathcal{Y}}\right)\ ,$$

and it is a general property of fundamental classes that under this map

$$
\begin{array}{ccc}
[\mathcal{Z}]_{\mathcal{X}} & \longrightarrow & [\mathcal{Z}]_{\mathcal{Y}} \\
\| & & \| \\
\begin{pmatrix}\text{fundamental class}\\ \text{of } \mathcal{Z} \text{ in } \mathcal{X}\end{pmatrix} & & \begin{pmatrix}\text{fundamental class}\\ \text{of } \mathcal{Z} \text{ in } \mathcal{Y}\end{pmatrix}\ .
\end{array}
$$

The argument now consists in tracing through the constructions of $\delta\nu_{\mathcal{Z},\mathcal{X}}$ and $\delta\nu_{\mathcal{Z},\mathcal{Y}}$ given in the Appendix, and verifying that, when this is done and the fundamental classes are fit into the exact hypercohomology sequence of (3.20) and related by the coboundary map (3.27), their infinitesimal invariants are related by (3.25).

Finally, the proof of the proposition may be completed by using the lemma together with the proposition in Section 3(a). Since the argument is entirely analogous to that given in [9], pages 83–85, and is just a matter of writing out the relations resulting from the hypercohomology sequence of (3.26) and standard duality, we will not present the details. This completes the proof of Theorem 1.

4 Proof of Theorem 2

(a) Let $\mathcal{X} \to S$ be a family $\{X_s\}_{s\in S}$ of smooth curves of genus g for which the Kodaira-Spencer mappings

$$\rho : T_s S \to H^1(\Theta_{X_s})$$

are surjective. Associated to $\mathcal{X} \to S$ is the family $\{Y_s\}_{s\in S}$ of surfaces $Y_s = X_s \times X_s$ given as

$$\begin{array}{l} \mathcal{Y} = \mathcal{X} \times_S \mathcal{X} \\ \downarrow \\ S\ . \end{array} \tag{4.1}$$

For each s we have defined the (rational equivalence class of) 0-cycle z_{K_s} by

$$z_{K_s} = D_s \times D_s - (2g-2)D_{s,\Delta}$$

where D_s is a divisor with $[D_s] = K_{X_s}$. Passing to a Zariski open set and finite covering of S if necessary, we may assume given

$$\mathcal{Z}_K \subset \mathcal{Y}$$

where

$$\mathcal{Z}_K \cdot Y_s \equiv z_{K_s}\ .$$

We want to show when $g \geqq 4$ that the associated infinitesimal invariant

$$\delta\nu_{\mathcal{Z}_K} \neq 0\ .$$

With our usual notations

$$\begin{cases} T = T_{s_0}S \\ V = H^0(\Omega^1_X) \\ U = (V\otimes V^*)\oplus(V^*\otimes V)\subset H^1(\Omega^1_Y) \end{cases}$$

where s_0 is a generic point of S and $X = X_{s_0}$,[3] we have for the value $\delta\nu_{K_X}$ of $\delta\nu_{\mathcal{Z}_K}$ at s_0 that

$$\delta z_{K_X} \in \Lambda^2 T^* \otimes \Lambda^2 V^* / \nabla(T^*\otimes U)\ .$$

Here, we are using the observation from Section 3 above that $\delta\nu_{K_X}$ lies in the subspace $\Lambda^2T^*\otimes\Lambda^2H^1(\mathcal{O}_X)$ of $\Lambda^2T^*\otimes H^2(\mathcal{O}_Y)$. By duality

$$\delta\nu_{K_X}\in\Big\{\ker\{\Lambda^2T\otimes\Lambda^2V\to T\otimes U\}\Big\}^*\ .$$

From the discussion in Section 2, we may use Schiffer variations to construct elements of $\ker\{\Lambda^2T\otimes\Lambda^2V\to T\otimes U\}$ as follows. Given distinct points $p,q\in X$ and differentials $\omega,\varphi\in H^0(\Omega^1_X(-p-q))$ we consider

$$\xi = \theta_p\wedge\theta_q\otimes\omega\wedge\varphi\in\Lambda^2T\otimes\Lambda^2V\ . \tag{4.2}$$

Here, θ_p and θ_q are elements of $H^1(\Theta_X)$ and, for simplicity of notation, we are omitting reference to the surjective map

$$T\to H^1(\Theta_X)\ .$$

Recall that θ_p is uniquely defined up to scaling, and that the scaling is fixed by a choice of local coordinate z with $z(p)=0$. As explained in Section 2, we may think of

$$\theta_p = \delta\left\{\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\right\}\ . \tag{4.3}$$

Locally, $\omega = f(z)dz$ with $f(0)=0$, and we have

$$\omega'(p) = f'(0)\ .$$

We note that θ_p and $\omega'(p)$ scale oppositely with z. A similar observation applies also to q, and therefore the quantity

$$\omega'(p)\varphi'(q)-\omega'(q)\varphi'(p)$$

[3] Proof analysis shows that $\delta\nu_{\mathcal{Z}_K}(s)\neq 0$ for any $s\in S$ such that $T_sS\to H^1(\Theta_{X_s})$ is surjective and X_s is non-hyperelliptic. Also, the $H^1(\Omega^1_X)\oplus H^1(\Omega^1_X)$ part of $H^1(\Omega^1_Y)$ does not enter into the calculation.

is intrinsically associated to ξ given by (4.2). To express this in a coordinate free manner, θ_p arises from a choice of a non-zero element of $\Theta_X(p)\mid_p$, and ω gives an element of $\Omega^1_X(-p)\mid_p$. Under the natural pairing

$$\langle\,,\rangle : \Omega^1_X(-p)\mid_p \otimes \Theta_X(p)\mid_p \to \mathbb{C}$$

we have

$$\langle \theta_p, \omega\rangle = \omega'(p)\ .$$

Finally, from the discussion in Section 2 we have

$$\xi \in \ker\{\Lambda^2 T \otimes \Lambda^2 V \to T \otimes U\}\ .$$

Our main calculation is given by the following

Theorem: *For ξ given by (4.2) we have*

$$\delta\nu_{K_X}(\theta_p \wedge \theta_q \otimes \omega \wedge \varphi) = \omega'(p)\varphi'(q) - \omega'(q)\varphi'(p)\ . \tag{4.4}$$

Assuming (4.4) we may complete the proof of Theorem 2 as follows: Let t_p, t_q be the tangent lines to the canonical curve $\varphi_{K_X}(X)$ at p, q respectively. Choose hyperplanes H_1, H_2 such that

$$\begin{cases} H_1 = 0 \ \text{ on } \ t_p \cup q \ \text{ but not on } \ t_p \cup t_q \\ H_2 = 0 \ \text{ on } \ t_q \cup p \ \text{ but not on } \ t_p \cup t_q\ . \end{cases}$$

Letting ω, φ be the corresponding 1-forms given by H_1, H_2 respectively, we have

$$\begin{cases} \omega'(p) = 0,\ \omega'(q) \neq 0 \\ \varphi'(p) \neq 0,\ \varphi'(q) = 0\ , \end{cases}$$

in which case $\delta\nu_{K_X}(\xi) \neq 0$ by (4.4). We can choose H_1, H_2 as above provided that

$$t_p \cap t_q = \emptyset\ .$$

However, for a non-degenerate embedded curve in $\mathbb{P}^r$ with $r \geqq 3$, the tangent lines at two general points do not meet. The condition $r \geqq 3$ for the canonical curve is $g \geqq 4$, thereby proving our result.

Heuristic: The quantity $\delta\nu_{K_X}(\xi)$ has the properties

(i) it is alternating in p, q and bilinear alternating in ω, φ;

(ii) it depends only on the "$1^{\rm st}$ order behaviour" of ω, φ near p, q (this is because a Schiffer variation θ_p leaves $X - \{p\}$ unchanged).

The only quantity that has the properties (i) and (ii) is a constant multiple of (4.4). This heuristic reasoning certainly suggests the result, *provided* that we know that the constant is non-zero. Even if we assume (ii) on the basis of geometric reasoning, a computation is required to estabish the latter point.

(b) For a line bundle $L \to X$, we denote by $L_i \to Y$ $(i = 1, 2)$ the line bundles on $X \times X$ induced by the projections, and by Δ the line bundle corresponding to the diagonal. Thinking of L_i and Δ as elements in $CH^1(Y)$ we have

$$z_L = L_1 \cdot L_2 - n\Delta \cdot L_1 \qquad (n = \deg L)$$

where the product is

$$CH^1(Y) \otimes CH^1(Y) \to CH^2(Y)$$

(clearly $\Delta \cdot L_1 = \Delta \cdot L_2$). Denoting by $\mathcal{L}_i \to \mathcal{Y}$ the line bundles induced over $\mathcal{Y} = \mathcal{X} \times_S \mathcal{X}$ by $\mathcal{L} \to \mathcal{X}$ and by $\tilde{\Delta}$ the line bundle given by the diagonals on $X_s \times X_s$, we have therefore

$$\mathcal{Z}_L = \mathcal{L}_1 \cdot \mathcal{L}_2 - n\widetilde{\Delta} \cdot \mathcal{L}_1 \ .$$

Denoting by $[\mathcal{J}]$ the Chern class in $\mathbb{H}^2\left(\Omega^{\geqq 1}_{\mathcal{Y}}\right)$ of a line bundle $\mathcal{J} \to \mathcal{Y}$, for the fundamental class $[\mathcal{Z}_L] \in \mathbb{H}^4\left(\Omega^{\geqq 2}_{\mathcal{Y}}\right)$ we have

$$[\mathcal{Z}_L] = [\mathcal{L}_1] \cup [\mathcal{L}_2] - n[\tilde{\Delta}] \cup [\mathcal{L}_1] \ . \tag{4.5}$$

The Leray filtration $F^i\mathbb{H}^*(\Omega^\bullet_{\mathcal{Y}})$ is induced by

$$\text{image}\left\{\Omega^i_S \otimes \Omega^{\bullet - i}_{\mathcal{Y}} \to \Omega^\bullet_{\mathcal{Y}}\right\}$$

(we omit the π^*'s), and

$$[\mathcal{Z}_L] \in F^2\mathbb{H}^4\left(\Omega^{\geqq 2}_{\mathcal{Y}}\right) \ .$$

The infinitesimal invariant

$$\delta\nu_{\mathcal{L}} \in \Omega^2_S \otimes R^2_\pi \mathcal{O}_{\mathcal{Y}} / \nabla\left(\Omega^1_S \otimes R^1_\pi \Omega^1_{\mathcal{Y}/S}\right)$$

is obtained from $[\mathcal{Z}_L]$ by localizing on S and using properties of the VHS associated to $\mathcal{Y} \to S$ (cf. the Appendix).

There seem to be two ways of computing $\delta\nu_{\mathcal{L}}$. One is geometric by choosing divisors D_s with $[D_s] = L_s$ and using the result in Section 3(b) above; this is the method used by [9]. The other is to express $\delta\nu_{\mathcal{L}}$ in terms of the Kodaira-Spencer classes in $H^1(\Sigma_{L_s})$; this is the method we followed in the "toy-model" example in Section 2 above. The computation we shall give is in some sense a mix of these two. The term $\omega'(p)$ is ultimately produced by

$$\mathrm{Res}_p\left(\frac{1}{z^2}\omega\right)$$

and the second order pole is produced from the transition functions of $\Sigma_{\mathcal{K}}$ by the term

$$\partial_z\theta_{p\alpha\beta}$$

in the off-diagonal spot. The calculation itself is carried out by selecting divisors $D_s \in |K_{X_s}|$ and using a more or less standard explicit relation between residues and duality.

Before turning to the formal computation we want to mention the following

Problem: *Let $\{Y_s\}_{s\in S}$ be a family of smooth algebraic surfaces and $E_s^j \to Y_s$ $(j = 1, \dots, m)$ a collection of families of rank-2 vector bundles such that if we set*

$$z_s = \sum_j n_j c_2(E_s^j) \in CH^2(Y_s)$$

where the n_j are integers, then

$$\begin{cases} \deg z_s = 0 \\ \mathrm{Alb}_{Y_s}(z_s) = 0\ . \end{cases}$$

The problem is to express the infinitesimal invariant $\delta\nu_z$ associated to the z_s in terms of the Kodaira-Spencer mappings

$$T_s \to H^1(\Sigma_{E_s^j})\ .$$

(c) We now turn to the formal calculation to prove (4.4). It is based on three principles:

(i) For any line bundle $\mathcal{J} \to \mathcal{Y}$ given by data as explained in Section 2 above, there is a canonical Čech representative for $c_1(\mathcal{J})$ in $H^1(\{U_\alpha\}, \Omega^1_{\mathcal{Y}})$. The "vertical" part — i.e., the image in $H^1(\{U_\alpha\}, \Omega^1_{\mathcal{Y}/S})$ — gives $c_1(\mathcal{J}_s)$. The "horizontal" part — which is well-defined only for the given data — contains the "$\mathcal{O}_{Y_s}$-part" of the Kodaira-Spencer class in $H^1(\sum_{J_s}) \otimes T_s^* S$ (recalling that the Kodaira-Spencer class gives the tangent to the family $\{J_s \to Y_s\}_{s\in S}$).

(ii) In (i) the "horizontal part" and "$\mathcal{O}_{Y_s}$-part" are not intrinsically defined for the line bundles $\mathcal{L}_1, \mathcal{L}_2$, and $\tilde{\Delta}$, but with the introduction of two pieces of auxilary data they are defined for the combination (4.5). The expression "auxilary data" is being used as in the toy model in Section 2; it refers to terms that are introduced to explicitly write a cocycle as a coboundary.

(iii) For $\mathcal{L} = \mathcal{K}$ and $\xi = \theta_p \wedge \theta_q \otimes \omega \wedge \varphi$ as above, the value $\delta\nu_{K_X}(\xi)$ may be calculated by residues, and when this is done the auxilary data in (ii) drops out and all that is left are the "principal parts" as explained above, and this is non-zero. Thus, although we have not computed $\delta\nu_{K_X}(\xi)$ for a general $\xi \in \ker\{\Lambda^2 T \otimes \Lambda^2 V \to T \otimes U\}$, we have computed enough to show that $\delta\nu_{K_X}$ is non-zero (we do not know how to compute $\delta\nu_{K_X}(\xi)$ for a general ξ).

Initially, we will for simplicity of notation assume that $\dim S = 1$ and denote by s a local uniformizing parameter on S. The extension to $\dim S = N$ with coordinates $s = (s_1, \ldots, s_N)$ will be obvious and will be introduced as needed.

Given data $\{U_\alpha, z_\alpha, f_{\alpha\beta}(z_\beta, s), \xi_{\alpha\beta}(z_\beta, s)\}$ for the situation

$$\begin{array}{c} \mathcal{L} \to \mathcal{X} \\ \quad\ \downarrow \\ \quad\ S \end{array} \tag{4.6}$$

we have in (2.9) given the transition data for $\sum_{\mathcal{L}}$; we note that there the two right hand entries in the top now give the Čech representative for $c_1(\mathcal{L})$.

For $\mathcal{Y} = \mathcal{X} \times_S \mathcal{X}$ as in (4.1), we will use the product coordinate data

$$\{U_\alpha \times U_\lambda, (z_\alpha, w_\lambda), (f_{\alpha\beta}(z_\beta, s), f_{\lambda\mu}(w_\mu, s))\}\ .$$

Then the data for (4.6) induces data for $\mathcal{L}_1 \to \mathcal{Y}$ and $\mathcal{L}_2 \to \mathcal{Y}$, and this in turn gives canonical Čech representatives for $[\mathcal{L}_1]$ and $[\mathcal{L}_2]$. We will write this out explicitly below.

The main remaining issue is to use the coordinate data for $\mathcal{X} \to S$ to canonically produce transition data for $\tilde{\Delta}$ and $\Sigma_{\tilde{\Delta}}$. For this we use the following construction due to Grothendieck: For any line bundle

$$J \to Z$$

over a smooth variety Z we have

$$\Sigma_J^* \cong J_1 \otimes J_2^{-1} \otimes \mathcal{O}_{Z\times Z}/\mathcal{I}^2_{\Delta_Z} \tag{4.7}$$

where $J_i = pr_i^* J$ and $\Delta_Z \subset Z \times Z$ is the diagonal with ideal $\mathcal{I}_{\Delta_Z}$. It is useful to verify (4.7), as this will help establish notation for what follows. We will assume that $\dim Z = 1$; the extension to the general case will be obvious. If $\{U_\alpha, z_\alpha, f_{\alpha\beta}(z_\beta)\}$ give coordinate data for Z and if $J \to Z$ has transition functions $\xi_{\alpha\beta}(z_\beta)$, then we cover a neighborhood of Δ_Z by $U_\alpha \times U_\alpha$ and relative to these open sets $J_1 \otimes J_2^{-1}$ has transition functions $\xi_{\alpha\beta}(z_\beta)\xi_{\alpha\beta}^{-1}(w_\beta)$. We write

$$\begin{aligned}
\frac{\xi_{\alpha\beta}(w+z-w)}{\xi_{\alpha\beta}(w)} &\equiv \frac{\xi_{\alpha\beta}(w)+(z-w)\partial_w\xi_{\alpha\beta}(w)}{\xi_{\alpha\beta}(w)} \mod (z-w)^2\\
&\equiv 1+\partial_w \log\xi_{\alpha\beta}(w)(z-w)\ .
\end{aligned}$$

We may give a section over $U_\alpha \times U_\alpha$ of the RHS of (4.7) by $A_\alpha(w_\alpha) + (z_\alpha - w_\alpha)B_\alpha(w_\alpha)$, and in $(U_\alpha \cap U_\beta) \times (U_\alpha \cap U_\beta)$

$$A_\alpha(w_\alpha)+(z_\alpha-w_\alpha)B_\alpha(w_\alpha) = \Big(1+(z_\beta-w_\beta)\partial_{w_\beta}\log\xi_{\alpha\beta}(w_\beta)\Big)\left(A_\beta(w_\beta)+(z_\beta-w_\beta)B_\beta(w_\beta)\right)\ ,$$

or more compactly

$$\begin{pmatrix} A_\alpha \\ (z_\alpha-w_\alpha)B_\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (z_\beta-w_\beta)\partial_{w_\beta}\log\xi_{\alpha\beta} & 1 \end{pmatrix}\begin{pmatrix} A_\beta \\ (z_\beta-w_\beta)B_\beta \end{pmatrix}\ . \tag{4.8}$$

Near Δ_Z

$$\frac{z_\alpha-w_\alpha}{z_\beta-w_\beta} = \frac{f_{\alpha\beta}(z_\beta)-f_{\alpha\beta}(w_\beta)}{z_\beta-w_\beta} \equiv \partial_{w_\beta}f_{\alpha\beta}(w_\beta) \mod (z_\beta-w_\beta)^2\ ,$$

from which it follows that $(z_\alpha - w_\alpha)B_\alpha(w_\alpha)$ is a section of $\Omega^1_{\Delta_Z} \cong \Omega^1_Z$, and then using w_α as coordinate on Δ_Z we may identify this term with $B_\alpha(w_\alpha)dw_\alpha$. Thus, (4.8) gives the transition data for the exact sequence

$$0 \to \Omega^1_Z \to \Sigma^*_J \to \mathcal{O}_Z \to 0\ ,$$

as claimed.

Next, we will show that the transition functions for Σ^*_Δ on $Y = X \times X$ are given in $(U_\alpha \cap U_\beta) \times (U_\alpha \cap U_\beta)$ by

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{dz_\beta}{z_\beta-w_\beta} - \frac{dz_\alpha}{z_\alpha-w_\alpha} & 1 & 0 \\ \frac{-dw_\beta}{z_\beta-w_\beta} + \frac{dw_\alpha}{z_\alpha-w_\alpha} & 0 & 1 \end{pmatrix}\ . \tag{4.9}$$

This has the following meaning: Transition functions for the exact sequence

$$0 \to \Omega^1_Y \to \Sigma^*_\Delta \to \mathcal{O}_Y \to 0$$

are given by

$$\begin{pmatrix} A_\alpha \\ B_\alpha dz_\alpha \\ C_\alpha dw_\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{dz_\beta}{z_\beta - w_\beta} - \frac{dz_\alpha}{z_\alpha - w_\alpha} & 1 & 0 \\ \frac{-dw_\beta}{z_\beta - w_\beta} + \frac{dw_\alpha}{z_\alpha - w_\alpha} & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\beta \\ B_\beta dz_\beta \\ C_\beta dw_\beta \end{pmatrix} .$$

To prove (4.9) we note that, relative to the covering $U_\alpha \times U_\alpha$ of the diagonal, the line bundle Δ has transition functions

$$\mu_{\alpha\beta} = \frac{z_\alpha - w_\alpha}{z_\beta - w_\beta} .$$

We use coordinates z_1, w_1, z_2, w_2 on $Y \times Y = X \times X \times X \times X$. Then $\Delta_1 \otimes \Delta_2^{-1}$ has transition functions

$$\frac{z_{1\alpha} - w_{1\alpha}}{z_{1\beta} - w_{1\beta}} \cdot \frac{z_{2\beta} - w_{2\beta}}{z_{2\alpha} - w_{2\alpha}} = \left(\frac{z_{2\beta} - w_{2\beta}}{z_{1\beta} - w_{1\beta}}\right) \left(\frac{z_{2\alpha} - w_{2\alpha}}{z_{1\alpha} - w_{1\alpha}}\right)^{-1} . \tag{4.10}$$

Since

$$\frac{z_{2\beta} - w_{2\beta}}{z_{1\beta} - w_{1\beta}} \equiv 1 + \frac{(z_{2\beta} - z_{1\beta}) - (w_{2\beta} - w_{1\beta})}{z_{1\beta} - w_{1\beta}} \tag{4.11}$$

modulo terms vanishing to 2^{nd}-order on the diagonal in $Y \times Y$, we may plug (4.11) into (4.10) to get that the transition functions of $\Delta_1 \otimes \Delta_2^{-1}$ are $\equiv$ to

$$\frac{1 + (z_{2\beta} - z_{1\beta}) - (w_{2\beta} - w_{1\beta})}{z_{1\beta} - w_{1\beta}} - \frac{(z_{2\alpha} - z_{1\alpha}) - (w_{2\alpha} - w_{1\alpha})}{z_{1\alpha} - w_{1\alpha}} . \tag{4.12}$$

Using the equation

$$A_\alpha + (z_{2\alpha} - z_{1\alpha})B_\alpha + (w_{2\alpha} - w_{1\alpha})C_\alpha \equiv (4.12) \times ((A_\beta + (z_{2\beta} - z_{1\beta})B_\beta + (w_{2\beta} - w_{1\beta})C_\beta))$$

we obtain

$$\begin{aligned} A_\alpha &= A_\beta \\ (z_{2\alpha} - z_{1\alpha})B_\alpha &= (z_{2\beta} - z_{1\beta})B_\beta + \left(\frac{z_{2\beta} - z_{1\beta}}{z_{1\beta} - w_{1\beta}} - \frac{z_{2\alpha} - z_{1\alpha}}{z_{1\alpha} - w_{1\alpha}}\right) A_\beta \\ (w_{2\alpha} - w_{1\alpha})C_\alpha &= (w_{2\beta} - w_{1\beta})C_\beta + \left(\frac{w_{2\alpha} - w_{1\alpha}}{z_{1\alpha} - w_{1\alpha}} - \frac{w_{2\beta} - w_{1\beta}}{z_{1\beta} - w_{1\beta}}\right) A_\beta . \end{aligned}$$

Then replacing $z_{2\alpha} - z_{1\alpha}$ modulo terms vanishing to 2^{nd}-order by dz_α, we obtain the transition matrix (4.9).

Now we want to extend the above calculation to the line bundle $\tilde{\Delta} \to \mathcal{Y}$. The new ingredient — in fact the one that makes this whole computation interesting — is that there is now s-dependence. Following the same procedure as above, we want to see how

$$A_\alpha + (z_{2\alpha} - z_{1\alpha})B_\alpha + (w_{2\alpha} - w_{1\alpha})C_\alpha + (s_2 - s_1)D_\alpha \tag{4.13}$$

changes coordinates. Recalling our notation

$$\begin{cases} j_{\alpha\beta} = \partial_{z_\beta} f_{\alpha\beta}(z_\beta, s) \\ \theta_{\alpha\beta} = \partial_s f_{\alpha\beta}(z_\beta, s) \end{cases}$$

we now have

$$\begin{aligned} z_{2\alpha} - z_{1\alpha} &= f_{\alpha\beta}(z_{2\beta}, s_2) - f_{\alpha\beta}(z_{1\beta}, s_1) \\ &\equiv j_{\alpha\beta}(z_{2\beta} - z_{1\beta}) + \theta_{\alpha\beta}(s_2 - s_1) \end{aligned}$$

and similarly for $w_{2\alpha} - w_{1\alpha}$. Setting $j_{1\alpha\beta} = j_{\alpha\beta}(z_\beta)$, $j_{2\alpha\beta} = j_{\alpha\beta}(w_\beta)$ and similarly for $\theta_{1\alpha\beta}$ and $\theta_{2\alpha\beta}$, it follows that the transition rules for the coefficients in (4.13) are

$$\begin{aligned} A_\alpha &= A_\beta \\ B_\alpha &= \left(\frac{1}{j_{1\alpha\beta}}\right) B_\beta + \left[\left(\frac{1}{(z_{1\beta} - w_{1\beta})} j_{1\alpha\beta}\right) - \left(\frac{1}{z_{1\alpha} - w_{1\alpha}}\right)\right] A_\beta \\ C_\alpha &= \left(\frac{1}{j_{2\alpha\beta}}\right) C_\beta - \left[\left(\frac{1}{(z_{1\beta} - w_{1\beta})} j_{2\alpha\beta}\right) - \left(\frac{1}{z_{1\alpha} - w_{1\alpha}}\right)\right] A_\beta \\ D_\alpha &= D_\beta - \left(\frac{\theta_{1\alpha\beta}}{j_{1\alpha\beta}}\right) B_\beta - \left(\frac{\theta_{2\alpha\beta}}{j_{2\alpha\beta}}\right) C_\beta \\ &\quad + \left[\frac{\left(\frac{\theta_{1\alpha\beta}}{j_{1\alpha\beta}}\right) - \left(\frac{\theta_{2\alpha\beta}}{j_{2\alpha\beta}}\right)}{(z_{2\beta} - w_{2\beta})}\right] A_\beta\ . \end{aligned}$$

Setting

$$\rho_{\alpha\beta}(z_\beta) = \theta_{\alpha\beta}(z_\beta)\partial/\partial z_\alpha \in Z^1(\{U_\alpha\}, \Theta_{\mathcal{X}/S})$$

we may rewrite the above equations along the diagonals in $\mathcal{Y} \times_S \mathcal{Y}$ as

$$\begin{aligned} A_\alpha &= A_\beta \\ B_\alpha dz_\alpha &= B_\beta dz_\beta + \left(\frac{dz_\beta}{z_{1\beta} - w_{1\beta}} - \frac{dz_\alpha}{z_{1\alpha} - w_{1\alpha}}\right) A_\beta \end{aligned}$$

$$\begin{aligned} C_\alpha dw_\alpha &= C_\beta dw_\beta - \left(\frac{dw_\beta}{z_{1\beta} - w_{1\beta}} - \frac{dw_\alpha}{z_{1\alpha} - w_{1\alpha}} \right) A_\beta \\ D_\alpha &= D_\beta - (\rho_{\alpha\beta}(z_\beta) \rfloor B_\beta dz_\beta) - (\rho_{\alpha\beta}(w_\beta) \rfloor C_\beta dw_\beta) \\ &+ \left[(\rho_{\alpha\beta}(z_\beta) + \rho_{\alpha\beta}(w_\beta)) \rfloor \frac{(dz_\beta - dw_\beta)}{z_\beta - w_\beta} \right] A_\beta \ . \end{aligned}$$

These may be summarized as

(4.14)

$$\begin{pmatrix} A_\alpha \\ B_\alpha dz_\alpha + C_\alpha dw_\alpha \\ D_\alpha ds \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{dz_\beta - dw_\beta}{z_\beta - w_\beta} - \frac{dz_\alpha - dw_\alpha}{z_\alpha - w_\alpha} & 1 & 0 \\ (\rho_{\alpha\beta}(z_\beta) + \rho_{\alpha\beta}(w_\beta)) \rfloor \frac{dz_\beta - dw_\beta}{z_\beta - w_\beta} & (\rho_{\alpha\beta}(z_\beta) + \rho_{\alpha\beta}(w_\beta)) & 1 \end{pmatrix} \begin{pmatrix} A_\beta \\ B_\beta dz_\beta + C_\beta dw_\beta \\ D_\beta ds \end{pmatrix} .$$

Symbolically, the RHS is

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline \text{Extension data for } \Sigma^*_{\tilde\Delta} & \multicolumn{2}{c}{\text{Transition}} \\ \cline{1-1} \text{Extra extension data for } \Sigma^*_{\tilde\Delta} & \multicolumn{2}{c}{\text{data for } \Omega^1_{\mathcal{Y}}} \end{array} \right) \begin{pmatrix} \mathcal{O}_{\mathcal{Y}} \\ \Omega^1_{\mathcal{Y}/S} \\ \Omega^1_S \otimes \mathcal{O}_{\mathcal{Y}} \end{pmatrix}$$

reflecting the exact sequences

$$\begin{cases} 0 \to \Omega^1_{\mathcal{Y}} \to \Sigma^*_{\tilde\Delta} \to \mathcal{O}_{\mathcal{Y}} \to 0 \\ 0 \to \Omega^1_S \otimes \mathcal{O}_{\mathcal{Y}} \to \Omega^1_{\mathcal{Y}} \to \Omega^1_{\mathcal{Y}/S} \to 0 \ . \end{cases}$$

We next set $\mathcal{K}_i = pr_i^* \mathcal{K}$ where

$$\begin{array}{ccc} \mathcal{K} & \to & \mathcal{X} \\ & & \downarrow \\ & & B \end{array}$$

is the family $\{K_{X_s} \to X_s\}_{s \in S}$ of canonical line bundles and

$$\begin{array}{ccc} & \mathcal{Y} = \mathcal{X} \times_S \mathcal{X} & \\ {}^{pr_1}\swarrow & & \searrow^{pr_2} \\ \mathcal{X} & & \mathcal{X} \end{array}$$

are the projections. In (2.9) we have given the transition data for $\Sigma_{\mathcal{L}} \to \mathcal{X}$ for a general line bundle $\mathcal{L} \to \mathcal{X}$. Taking $\mathcal{L} = \mathcal{K}$, the extension data part of the transition data for $\Sigma^*_{\mathcal{K}}$ is

$$\begin{pmatrix} \left(\partial_{z_\beta} \log j_{\alpha\beta}(z_\beta, s)\right) dz_\beta \\ \left(\frac{\partial_s j_{\alpha\beta}(z_\beta,s)}{j_{\alpha\beta}(z_\beta,s)}\right) ds \end{pmatrix}$$

which now taking $s = (s_1, \ldots, s_N)$ we may write as

$$\begin{pmatrix} \left(\partial_{z_\beta} \log j_{\alpha\beta}\right) dz_\beta \\ \sum_i \left(\partial_{s_i} \log j_{\alpha\beta}\right) ds_i \end{pmatrix} .$$

Pulling minus this[4] back to $\mathcal{Y} = \mathcal{X} \times_S \mathcal{X}$ under the two projections gives Čech representatives for $[\mathcal{K}_1]$ and $[\mathcal{K}_2]$.

Here we need to remark about the open coverings of $\mathcal{Y}$. These transition functions and extension data are relative to $\{U_\alpha \times \mathcal{X}\}$, whereas those for $\Sigma^*_{\tilde{\Delta}}$ above are relative to $\{U_\alpha \times U_\alpha\}$ — as will be seen below, the latter will be all that is needed for the calculation of (4.5) when $\mathcal{L} = \mathcal{K}$.

At this stage, in terms of the data $\{U_\alpha, z_\alpha, f_{\alpha\beta}(z_\beta, s)\}$ giving $\mathcal{X} \to S$ we have canonical expressions for Čech cocycles representing each of the terms in (4.5) when $\mathcal{L} = \mathcal{K}$. The strategy for completing the calculation establishing (4.4) is this:

(i) Write each of the cocycles as a Čech coboundary of a cochain with poles — this will enable duality pairings to be computed by residues;

(ii) Assuming that $\dim S = 2$ and that $T_{s_0}S$ is spanned by Schiffer variations θ_p and θ_q, we may lift $\omega, \varphi \in H^0(\Omega^1_X)$ to $\tilde{\omega}, \tilde{\varphi} \in H^0(\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_X)$. Then

(4.15) $$\underbrace{\left([\mathcal{K}_1] \cup [\mathcal{K}_2] - (2g-2)[\tilde{\Delta}] \cup [\mathcal{K}_1]\right)} \wedge (\tilde{\omega}_1 \wedge \tilde{\varphi}_2 - \tilde{\omega}_2 \wedge \tilde{\varphi}_1) \in H^2(\Omega^4_{\mathcal{Y}} \otimes \mathcal{O}_Y) ,$$

which maps naturally to

(4.16) $$H^2(\Omega^2_{\mathcal{Y}/S} \otimes \Omega^2_S \otimes \mathcal{O}_Y) \cong \Lambda^2 T^* \otimes H^2(\Omega^2_Y) .$$

Using the fact that the term over the braces lies in the second level in the Leray filtration, and therefore in cohomology lies in

$$\Omega^2_S \otimes H^2(\mathcal{O}_{\mathcal{Y}}) ,$$

[4]Here we note the minus sign in (2.9)

we infer that (4.15) maps to

$$\delta\nu_{K_X}(\theta_p \wedge \theta_q \otimes \omega \wedge \varphi)\ ;^{5}$$

(iii) Finally, we use (i) to explicitly compute (4.15). The calculation will be based on the following

Observation: If D_1, D_2 are smooth divisors meeting transversely on Y, then the cohomology associated to

$$0 \to \Omega_Y^2 \to \bigoplus_{i=1}^{2} \Omega_Y^2 \otimes \mathcal{O}_Y(D_i) \to \Omega_Y^2 \otimes \mathcal{O}_Y(D_1 + D_2) \to \Omega_Y^2 \otimes \mathcal{O}_{D_1\cap D_2}(D_2 + D_2) \to 0$$

takes

$$H^0\left(\Omega_Y^2 \otimes \mathcal{O}_{D_1\cap D_2}(D_1 + D_2)\right) \to H^2(\Omega_Y^2)$$

by

$$\frac{\omega}{f_1 f_2} \to \sum_{y\in D_1\cap D_2} \operatorname{Res}_y\left(\frac{\omega}{f_1 f_2}\right)\ .$$

To carry out the argument, it will simplify the notation if we set

$$\theta_1 = \theta_p, \quad \theta_2 = \theta_q$$

$$\omega_1 = \omega, \quad \omega_2 = \varphi$$

and give ω_i explicitly by

$$\omega_i = \{h_{i\alpha}(z_\alpha)dz_\alpha\}\ .$$

[5] Alternatively,

$$\Omega_S^2 \otimes H^2(\mathcal{O}_y) \hookrightarrow H^2(\Omega_y^2 \otimes \mathcal{O}_y)$$

induces a commutative diagram

^2_S H ^{2}(O $_y$) H ^{2}(^2_Y O $_y$) – H ^{2}(^2_Y O $_y$) H ^{0}(^2_Y O $_y$)

^2_S H ^{2}(^2_Y O $_y$) H ^{2}(^2_Y O $_y$)

^2_Y H ^{2}(^2_Y O $_y$)

Since

$$([\mathcal{K}_1] \cup [\mathcal{K}_2] - [\widetilde{\Delta}] \cup [\mathcal{K}_1])\ |_y \in \Omega_S^2 \otimes H^2(\mathcal{O}_y)$$

we may compute how it pairs with $\omega_1 \wedge \varphi_2 - \omega_2 \wedge \varphi_1$ by wedging with $\tilde{\omega}_1 \wedge \tilde{\varphi}_2 - \tilde{\omega}_2 \wedge \tilde{\varphi}_1$.

Then since $\theta_i(\omega_j) = 0$ in $H^1(\mathcal{O}_X)$ we have

$$\theta_{i\alpha\beta}(z_\beta)h_{j\beta}(z_\beta) = k_{ij\alpha}(z_\alpha) - k_{ij\beta}(z_\beta) \tag{4.17}$$

where $\{k_{ij\alpha}\} \in C^1(\{U_\alpha\}, \mathcal{O}_X)$. From our calculation of the transition functions for $\mathcal{X} \to S$ and

$$\begin{pmatrix} h_{j\alpha}dz_\alpha \\ k_{1j\alpha}ds_1 + k_{2j\alpha}ds_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho_{1\alpha\beta}ds_1 + \rho_{2\alpha\beta}ds_2 & 1 \end{pmatrix} \begin{pmatrix} h_{j\beta}dz_\beta \\ k_{1j\beta}ds_1 + k_{2j\beta}ds_2 \end{pmatrix}$$

we infer that

$$\begin{pmatrix} h_{j\alpha}dz_\alpha \\ k_{1j\alpha}ds_1 + k_{2j\alpha}ds_2 \end{pmatrix} =: \tilde{\omega}_{j\alpha}$$

represents a lifting of ω_j to $\tilde{\omega}_j \in H^0(\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_X)$.

We will now carry out step (i). Recalling that the Čech expression for $[\widetilde{\Delta}] \in H^1(\Omega^1_{\mathcal{Y}})$ is (with the above understanding about open coverings)

$$-\begin{pmatrix} \frac{dz_\beta - dw_\beta}{z_\beta - w_\beta} - \frac{dz_\alpha - dw_\alpha}{z_\alpha - w_\alpha} \\ \sum_i \left(\rho_{i\alpha\beta}(z_\beta) \ + \rho_{i\alpha\beta}(w_\beta)\right) \rfloor \left(\frac{dz_\beta - dw_\beta}{z_\beta - w_\beta}\right) ds_i \end{pmatrix}$$

(where we recall that $\rho_{i\alpha\beta}(z_\beta) = \partial_{s_i} f_{\alpha\beta}(z_\beta, s)\partial/\partial z_\alpha$, etc.) we observe that

$$\delta\left\{\begin{pmatrix} \frac{dz_\alpha - dw_\alpha}{z_\alpha - w_\alpha} \\ 0 \end{pmatrix}\right\}$$
$$= \begin{pmatrix} \frac{dz_\alpha - dw_\alpha}{z_\alpha - w_\alpha} \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ \sum_i \left(\rho_{i\alpha\beta}(z_\beta) \ + \rho_{i\alpha\beta}(w_\beta)\right) ds_i & 1 \end{pmatrix} \begin{pmatrix} \frac{dz_\beta - dw_\beta}{z_\beta - w_\beta} \\ 0 \end{pmatrix} .$$

Thus

$$[\tilde{\Delta}] = -\delta \begin{pmatrix} \frac{dz_\alpha - dw_\alpha}{z_\alpha - w_\alpha} \\ 0 \end{pmatrix} . \tag{4.18}$$

Now let $\varphi = \{l_\alpha(z_\alpha, s)dz_\alpha\} \in H^0(\Omega^1_{\mathcal{X}/S})$ be chosen with divisor

$$\begin{cases} (\varphi) = D = \sum_\lambda r_\lambda \\ p, q \notin (\varphi) \ . \end{cases}$$

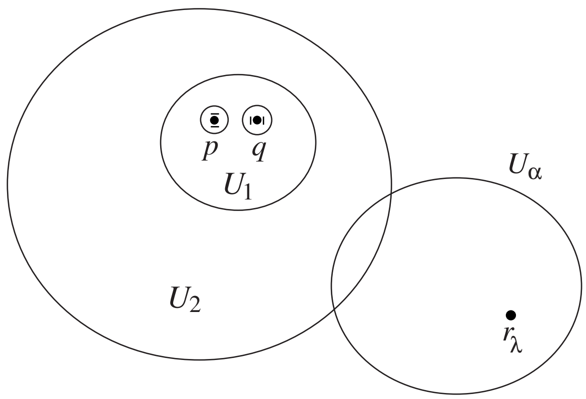
Then

$$[\mathcal{K}_1] = \delta \begin{pmatrix} \partial_{z_\alpha} \log l_\alpha(z_\alpha, s)dz_\alpha \\ \sum_i \partial_{s_i} \log l_\alpha(z_\alpha, s)ds_i \end{pmatrix} , \tag{4.19}$$

where the RHS is pulled back to $\mathcal{Y} = \mathcal{X} \times_S \mathcal{X}$ under the first projection, and similarly for $[\mathcal{K}_2]$ with w_α replacing z_α.

Next, to simplify the calculation and make explicit use of our assumption of Schiffer variations we will choose our covering of X as follows: U_1 will be a disc with coordinate z such that $p, q \in U_1$. U_2 will be a larger disc with small discs around p, q removed, and the remaining U_2 $(\alpha \geqq 2)$ will satisfy

$$\begin{cases} U_\alpha \cap U_1 = \emptyset \\ r_\lambda \in U_{\alpha(\lambda)} \qquad (\alpha(\lambda) \geqq 2) \end{cases} .$$



For $\theta_{1\alpha\beta}$ at $s = s_0$ we will have

$$\theta_{1\alpha\beta} = \delta\left\{\left(\frac{1}{z - z(p)}\right)\right\} =: \delta\{\sigma_\alpha\}$$

which is explicitly given by

$$\begin{cases} \theta_{112} = \frac{1}{z - z(p)} \\ \text{all other } \theta_{1\alpha\beta} = 0 . \end{cases}$$

There is a similar expression for $\theta_{2\alpha\beta}$ with q replacing p. Then in (4.17) we may take, e.g.,

$$k_{11\alpha} = \sigma_\alpha h_{1\alpha} .$$

It follows that all

(4.20) $$k_{ij\alpha}(r_\lambda) = 0 .$$

We will now carry out step (ii). Setting

$$\tilde{\omega}_{ij} = pr_i^* \tilde{\omega}_j$$

and using the general observation above together with (4.20) we have

$$[\mathcal{K}_1] \wedge \tilde{\omega}_{11} \wedge [\tilde{\Delta}] \wedge \tilde{\omega}_{22} =$$

$$\sum_\lambda \mathrm{Res}_{(r_\lambda, r_\lambda)} \left(\partial_{z_\alpha} \log l_\alpha(z_\alpha, s) dz_\alpha + \sum_i \partial_{s_i} \log l_\alpha(z_\alpha, s) ds_i \right) \wedge$$
$$(h_{1\alpha}(z_\alpha, s) dz_\alpha) \wedge \left(\frac{dz_\alpha - dw_\alpha}{z_\alpha - w_\alpha} \right) \wedge (h_{2\alpha}(w_\alpha, s) dw_\alpha) \Big] = 0$$

since there are no $ds_1 \wedge ds_2$ terms.

Note: The only possible $ds_1 \wedge ds_2$ terms would arise from expressions such as

$$\left(\sum_i \partial_{s_i} l_\alpha ds_i \right) \wedge \left(\sum_j k_{j2\alpha} ds_j \right) .$$

But since we are evaluating at (r_λ, r_λ) these are zero, by (4.20). The *philisophical* reason they are out is that the $k_{ji\alpha}$ represent "auxilary data", and for Schiffer variations the auxilary data does not enter into the final expressions for the same reason as in the "toy model" discussed in section 2. Intuitively, using Schiffer variations localizes the problem at p and q, and since the auxilary data is not uniquely specified by the given data its value at any point does not have intrinsic meaning.

We now turn to the evaluation of the "principal part" term

$$[\mathcal{K}_1] \wedge \tilde{\omega}_{11} \wedge [\mathcal{K}_2] \wedge \tilde{\omega}_{22} . \tag{4.21}$$

As above the $k_{ij\alpha}$ terms drop out and (4.21) is

$$\sum_{\lambda, i} \mathrm{Res}_{r_\lambda} \left[\left(\partial_{s_i} \log l_\alpha(z_\alpha, s) \right) h_{1\alpha}(z_\alpha, s) dz_\alpha \right] ds_i \wedge$$
$$\sum_{\mu, j} \mathrm{Res}_{r_\mu} \Big[(\partial_{s_j} \log l_\alpha(w_\alpha, s) h_{2\alpha}(w_\alpha, s)) dw_\alpha \Big] ds_j .$$

The proof will be complete if we show:

$$\sum_\lambda \mathrm{Res}_{r_\lambda} \left[\left(\partial_{s_1} \log l_\alpha(z_\alpha, s) \right) h_{1\alpha}(z_\alpha, s) dz_\alpha \right] = \omega_1'(p) . \tag{4.22}$$

Proof of (4.22): We have

$$l_\alpha(f_{\alpha\beta}(z_\beta, s), s) = \partial_{z_\beta} f_{\alpha\beta}(z_\beta, s)^{-1} l_\beta(z_\beta, s)$$

which gives

$$\partial_{s_1} \log l_\alpha = \partial_{s_1} \log l_\beta + \partial_{z_\alpha} \log l_\alpha \theta_{1\alpha\beta} - \frac{\partial_{z_\beta}\theta_{1\alpha\beta}}{\partial_{z_\beta} f_{\alpha\beta}} \ . \tag{4.23}$$

Near r_λ we have $\theta_{1\alpha\beta} = 0$ so that

$$\mathrm{Res}_{r_\lambda} \left(\omega_1 \partial_{s_1} \log l_\alpha\right) = \mathrm{Res}_{r_\lambda} \left(\omega_1 \partial_{s_1} \log l_\beta\right)$$

as expected. Now we may consider

$$(\omega_1 \partial_{s_1} \log l_\alpha) \in H^0\left(\Omega^1_X(D) \mid_D\right) \ ,$$

and by (4.23) under the cohomology sequence associated to

$$0 \to \Omega^1_X \to \Omega^1_X(D) \to \Omega^1_X(D) \mid_D \to 0$$

it maps to

$$\omega_1 \left(\partial_{z_\alpha} \log l_\alpha \theta_{1\alpha\beta} - \frac{\partial_{z_\beta}\theta_{1\alpha\beta}}{\partial_{z_\beta} f_{\alpha\beta}}\right) \in Z^1\left(\{U_\alpha\}, U^1_X\right) \to H^1(\Omega^1_X) \ . \tag{4.24}$$

Call the term in parentheses $\tau_{\alpha\beta}$; it is holomorphic since $\theta_{1\alpha\beta}$ vanishes at the zeroes of l_α. By our choice of open covering and construction of $\theta_{1\alpha\beta}$, all $\tau_{\alpha\beta} = 0$ except for τ_{12} and

$$\tau_{12} = (\partial_{z_1} \log l_1) \left(\frac{1}{z_1 - p}\right) + \frac{1}{(z_1 - p)^2}$$

since $f_{12} \equiv 1$. The $1/(z_1 - p)^2$ term comes from $\partial_{z_1}\theta_{112}$ — this is the crucial term, whose intuitive origin was discussed in Section 2. Now since $\omega_1(p) = 0$

$$\omega_1 = (a(z_1 - p) + \cdots)\, dz_1$$

where

$$a = \omega_1'(p) \ .$$

Then near p

$$\omega_1 \tau_{12} = \frac{\omega_1'(p) dz_1}{z_1 - p} + \text{holomorphic terms} \ ,$$

and it follows that (4.24) evaluates to $\omega_1'(p)$ in $H^1(\Omega^1_X) \cong \mathbb{C}$. □

Appendix: Definition of the infinitesimal invariants

(a) Classically, normal functions were introduced by Poincaré to study algebraic 1-cycles on an algebraic surface. The construction extends naturally to a situation

$$\text{(A.1)} \qquad \begin{array}{ccc} \mathcal{Z} & \subset & \mathcal{Y} \\ & & \downarrow_{\pi} \\ & & S\ . \end{array}$$

Here, $\mathcal{Y} \xrightarrow{\pi} S$ is a smooth family of projective algebraic varieties $\{Y_s\}_{s \in S}$, and $\mathcal{Z} \subset \mathcal{Y}$ is a codimension-p algebraic cycle such that $Z_s = \mathcal{Z} \cdot Y_s$ is a codimension-p cycle on each Y_s that is homologous to zero there. In this case the Abel-Jacobi image

$$\mathrm{AJ}_{Y_s}(Z_s) \in J^p(Y_s)$$

is defined, where $J^p(Y_s)$ is the p^{th} intermediate Jacobian of Y_s. Denoting by

$$\text{(A.2)} \qquad \mathcal{J} \to S$$

the family of intermediate Jacobians, there is defined a normal function $\nu_{\mathcal{Z}}$ by

$$\nu_{\mathcal{Z}}(s) = \mathrm{AJ}_{Y_s}(Z_s)\ .$$

In terms of the variation of Hodge structure[6] $\{H_{\mathbb{Z}}, \mathcal{F}^p, \nabla\}$ associated to (A.1), we have

$$J^p(Y_s) = \mathcal{F}^p_s \backslash \mathcal{H}_s / \mathcal{H}_{\mathbb{Z},s}\ .$$

In terms of any local lifting $\tilde{\nu}$ of ν to $\mathcal{H}$, we have

$$\text{(A.3)} \qquad \nabla \tilde{\nu} \in \Omega^1_S \otimes \mathcal{F}^{p-1}\ .$$

By definition, a *normal function* is any holomorphic section ν of (A.2) such that (A.3) is satisfied. Setting

$$\mathcal{H}^{r,s} = \mathcal{F}^r / \mathcal{F}^{r-1} \qquad (r + s = 2p - 1)$$

[6] Here, $\mathcal{H}_{\mathbb{Z}} = R^{2p-1}_{\pi}\mathbb{Z}$ and $\mathcal{H} = \mathcal{O}_S \otimes R^{2p-1}_{\pi}\mathbb{C}$ with Gauss-Manin connection

$$\nabla : \mathcal{H} \to \Omega^1_S \otimes \mathcal{H}\ ,$$

and $\mathcal{F}^p \subset \mathcal{H}$ is the Hodge filtration which satisfies

$$\nabla \mathcal{F}^p \subset \Omega^1_S \otimes \mathcal{F}^{p-1}\ .$$

there is an induced $\mathcal{O}_S$-linear mapping

$$\nabla : \mathcal{H}^{r,s} \to \Omega^1_S \otimes \mathcal{H}^{r-1,s+1} \tag{A.4}$$

that will be basic in what follows.

For a normal function ν the associated *infinitesimal invariant*

$$\delta\nu \in \Omega^1_S \otimes \mathcal{H}^{p-1,p}$$

is defined to be the section induced by $\nabla\tilde{\nu}$ for any local lifting $\tilde{\nu}$ of ν. Although $\nu_{\mathcal{Z}}$ is transcendental — i.e., in one form or another it involves integration — $\delta\nu_{\mathcal{Z}}$ is algebraic and in some cases $\delta\nu_{\mathcal{Z}}(s)$ may be interpreted geometrically in terms of the 1^{st}-order variation of Z_s. In general, $\delta\nu$ was introduced in an attempt to associate geometric invariants to ν.

(b) Let Y be a smooth variety. Associated to a codimension-p algebraic cycle Z one expects — according to Beilinson — to be able to successively associate a sequence of Hodge-theoretic invariants

$$\psi_0(Z), \psi_1(Z), \dots, \psi_p(Z)$$

where $\psi_i(Z)$ is defined only if $\psi_0(Z) = \cdots = \psi_{i-1}(Z) = 0$, and where the vanishing of all of the $\psi_i(Z)$ is equivalent, modulo torsion, to Z being rationally equivalent to zero. The first two have been defined: $\psi_0(Z)$ is the fundamental class of Z, and if $\psi_0(Z) = 0$ then $\psi_1(Z) \in J^p(Y)$ is the Abel-Jacobi image of Z. Now although the higher ψ_i's have not yet been defined in general (however, see [Green]), it is in some cases possible to define their "1^{st} variation." More precisely, if we have a family as in (A.1), and if

$$\psi_0(Z_s) = \cdots = \psi_{i-1}(Z_s) = 0$$

for all $s \in S$, then one might expect the $\psi_i(Z_s)$ to give something like a normal function

$$\nu^i_{\mathcal{Z}}(s) \in J^p_i(Y_s)\ .$$

Although this definition has not been given, one may succesively define invariants

$$\delta\nu^k_{\mathcal{Z}} \in H^m\left(\Omega^k_S \otimes \mathcal{H}^{p-k,p+k-m}, \nabla\right)$$

that extend the definition of $\delta\nu^1_{\mathcal{Z}}$ and have the property that

$$\delta\nu^k_{\mathcal{Z}} \neq 0 \Rightarrow \left\{ \begin{array}{l} z_s \text{ is non-torsion in } CH^p(Y_s) \\ \text{for general } s \end{array} \right\}\ .$$

For S affine we have

$$\delta\nu^k_{\mathcal{Z}} \in H^0\left(\Omega^k_S \otimes \mathcal{H}^{p-k,p+k}, \nabla\right) .$$

In the situation of interest to us when z_s is a family of 0-cycles or a family of algebraic surfaces Y_s,

$$\delta\nu^1_{\mathcal{Z}} \in \Omega^1_S \otimes \mathcal{H}^{1,2}/\nabla\mathcal{H}^{2,1}$$

turns out to be (see below) the infinitesimal invariant associated to the normal function

$$s \to \mathrm{Alb}_{Y_s}(z_s) .$$

The second, non-classical invariant

$$\delta\nu^2_{\mathcal{Z}} \in \Omega^2_S \otimes \mathcal{H}^{0,2}/\nabla\mathcal{H}^{1,1}$$

is what is of interest to us, *especially* computing it in cases of geometric interest.

The invariants referred to above have been around in various guises for some time, dating to the definition of the arithmetic cycle class by Grothendieck and Deligne and, from a completely different perspective the paper by Mumford (where he notes the the invariant he used was already discussed by Severi as early as 1936). More recently, special cases of it have appeared in the work of [8], [2], the Banff lectures by Green [4], and the work of the Japanese School (Asakura, S. Saito [6], M. Saito [7]). The use of these invariants in geometric problems that has been most influential in the present paper are in [5] and [9].

We shall now define $\delta\nu_{\mathcal{Z}} = \delta\nu^2_{\mathcal{Z}}$ in the situation (A.1) where $\{Y_s\}_{s\in S}$ is a family of smooth surfaces and $\mathcal{Z}\cdot Y_s = z_s$ is a family of 0-cycles with

$$\text{(A.5)} \qquad \begin{cases} \deg z_s = 0 \\ \mathrm{Alb}_{Y_s}(z_s) = 0 . \end{cases}$$

The construction is based on four things:

(i) $\mathcal{Z}$ defines a fundamental class

$$[\mathcal{Z}] \in \ \mathrm{Image}\ \{\mathbb{H}^2(\Omega^{\geqq 2}_{\mathcal{Y}}) \to \mathbb{H}^2(\Omega^\bullet_{\mathcal{Y}})\} ;$$

(ii) the Leray spectral sequence for $\mathcal{Y} \to S$ degenerates at E_2, and under the assumption (A.5)

$$[\mathcal{Z}] \in F^2H^4(\mathcal{Y}, \mathbb{C})$$

where F^i is the Leray filtration;

(iii) $\mathbb{H}^*(\Omega^\bullet_{\mathcal{Y}})$ computes $H^*(\mathcal{Y},\mathbb{C})$ and $\mathbb{R}^*_\pi(\Omega^\bullet_{\mathcal{Y}/S})$ computes $R^*_\pi\mathbb{C}$; and

(iv) the standard properties of the variation of Hodge structure (VHS) associated to $\mathcal{Y} \xrightarrow{\pi} S$.

To begin with, the Leray filtration is induced on hypercohomology by

$$\text{image}\left\{\Omega^i_S \otimes \Omega^{\bullet-i}_{\mathcal{Y}/S} \to \Omega^\bullet_{\mathcal{Y}}\right\}$$

whose associated graded is

$$\mathrm{Gr}^i\Omega^\bullet_{\mathcal{Y}} \cong \Omega^i_S \otimes \Omega^{\bullet-i}_{\mathcal{Y}/S} .$$

The E_1-term of the Leray spectral sequence is

$$\longrightarrow \Omega^q_S \otimes \mathbb{R}^p_\pi\left(\Omega^\bullet_{\mathcal{Y}/S}\right) \xrightarrow{\nabla} \Omega^{q+1}_S \otimes \mathbb{R}^p_\pi\left(\Omega^\bullet_{\mathcal{Y}/S}\right) \longrightarrow \quad . \tag{A.6}$$

The Hodge filtration on the $\mathbb{R}^p_\pi(\Omega^\bullet_{\mathcal{Y}/S})$ is induced by the subcomplexes

$$\Omega^{\geqq p}_{\mathcal{Y}/S} \subset \Omega^\bullet_{\mathcal{Y}/S} ,$$

and the spectral sequence induced by the Hodge filtration degenerates at E_1. Passing to the associated graded of the Hodge filtration in (A.6) gives the complex

$$\longrightarrow \mathcal{H}^{p,r} \otimes \Omega^q_S \xrightarrow{\nabla} \mathcal{H}^{p-1,r+1} \otimes \Omega^{q+1}_S \longrightarrow \tag{A.7}$$

where

$$\mathcal{H}^{p,r} \cong R^r_\pi\left(\Omega^p_{\mathcal{Y}/S}\right) .$$

Since our considerations are local in the base, we may take S to be affine. The fundamental class of $\mathcal{Z}$ defines

$$[\mathcal{Z}]^0 \in H^0(S, R^4_\pi\mathbb{C}) .$$

If this is zero then it defines

$$[\mathcal{Z}]^1 \in H^1(S, R^3_\pi\mathbb{C}) ,$$

and if this is zero then we have

$$[\mathcal{Z}]^2 \in H^2(S, R^2_\pi\mathbb{C}) .$$

From

$$\deg z_s = 0,$$

we infer that $[\mathcal{Z}]^0 = 0$. Then $[\mathcal{Z}]^1$ defines a class in the cohomology of the complex

$$\mathcal{O}_S \otimes R^3_\pi\mathbb{C} \xrightarrow{\nabla} \Omega^1_S \otimes R^3_\pi\mathbb{C} \xrightarrow{\nabla} \Omega^2_S \otimes R^3_\pi\mathbb{C} .$$

Since also $[\mathcal{Z}] \in \mathbb{H}^2(\Omega^{\geqq 2}_{\mathcal{Y}})$ we infer from (A.6) that $[\mathcal{Z}]^1$ defines a class in the cohomology of

$$\mathbb{R}^3_\pi \left(\Omega^{\geqq 2}_{\mathcal{Y}/S}\right) \xrightarrow{\nabla} \Omega^1_S \otimes \mathbb{R}^3_\pi \left(\Omega^{\geqq 1}_{\mathcal{Y}/S}\right) \xrightarrow{\nabla} \Omega^2_S \otimes \mathbb{R}^3_\pi \left(\Omega^{\geqq 0}_{\mathcal{Y}/S}\right) .$$

Finally, passing to the quotient by the Hodge filtration we see that $[\mathcal{Z}]^1$ defines

$$\delta\nu^1_{\mathcal{Z}} \in \Omega^1_S \otimes \mathcal{H}^{1,2}/\nabla\mathcal{H}^{2,1} .$$

Proposition: (i) $\delta\nu^1_{\mathcal{Z}}$ *is the infinitesimal invariant of the normal function*

$$s \to \mathrm{Alb}_{Y_s}(z_s) .$$

(ii) *If* $\mathrm{Alb}_{Y_s}(z_s) = 0$*, then* $[\mathcal{Z}]^1 = 0$.

Assuming the proposition, if (A.5) is satisfied then $[\mathcal{Z}]^2$ defines a cohomology class in the complex

$$\longrightarrow \Omega^1_S \otimes \mathbb{R}^2_\pi \left(\Omega^{\geqq 1}_{\mathcal{Y}/S}\right) \xrightarrow{\nabla} \Omega^2_S \otimes \mathbb{R}^2_\pi \left(\Omega^{\bullet}_{\mathcal{Y}/S}\right) \xrightarrow{\nabla} \Omega^3_S \otimes \mathbb{R}^2_\pi \left(\Omega^{\bullet}_{\mathcal{Y}/S}\right) \longrightarrow .$$

Passing to the quotient by the Hodge filtration as in (A.7) we see that $[\mathcal{Z}]^2$ defines

$$\delta\nu^2_{\mathcal{Z}} \in \Omega^2_S \otimes \mathcal{H}^{0,2}/\nabla \left(\Omega^1_S \otimes \mathcal{H}^{1,1}\right) .$$

Definition: *For the situation (A.1) where* $\mathcal{Y} = \{Y_s\}_{s\in S}$ *is a family of smooth surfaces and* $z_s \in Z^2(Y_s)$ *is a family of* 0*-cycles satisfying (A.5), we define the infinitesimal invariant*

$$\delta\nu_{\mathcal{Z}} = \delta\nu^2_{\mathcal{Z}} .$$

A special case is when

$$\mathcal{Y} = Y \times S$$

is a product. Then we do not need to assume that $[\mathcal{Z}]^0 = [\mathcal{Z}]^1 = 0$ to define $[\mathcal{Z}]^2$ as a Künneth component of $[\mathcal{Z}]$, and since the complex (A.7) is trivial we have

$$\delta\nu_{\mathcal{Z}} \in \Omega^2_S \otimes H^2(\mathcal{O}_Y) \cong \mathrm{Hom}\left(H^0(\Omega^2_Y), \Omega^2_S\right) .$$

In fact it may be seen that

$$\delta\nu_{\mathcal{Z}} = \mathrm{Tr}_{\mathcal{Z}}$$

is just the trace map

$$H^0(\Omega^2_Y) \to \Omega^2_S$$

given by

$$\mathrm{Tr}_{\mathcal{Z}}\omega = \pi_*(\omega \mid_{\mathcal{Z}}) .$$

Thus, $\delta\nu_{\mathcal{Z}}$ may be thought of as an extension to variable families of surfaces of the construction originally used by Mumford when he showed that $\dim CH^2(Y) = \infty$ if $H^0(\Omega^2_Y) \neq 0$ (cf. [9]).

Proof of the proposition: (i) There is an obvious formulation of the first part of the proposition for a family $\mathcal{D} = \{D_s\}_{s\in S}$ of divisors on a family of smooth algebraic curves

$$\{X_s\}_{s\in S} \ .$$

Moreover, if $X_s \subset Y_s$ is an ample curve then

$$J(X_s) \to \mathrm{Alb}(Y_s) \tag{A.8}$$

is surjective, and if D_s is chosen so that for each $s \in S$

$$D_s \equiv z_s$$

where $\equiv$ denotes rational equivalence, then under the mapping (A.8)

$$\nu_{\mathcal{D}} \to \nu_{\mathcal{Z}}$$

and

$$\delta\nu_{\mathcal{D}} \to \delta\nu_{\mathcal{Z}} \ .$$

Thus it will suffice to prove (i) for the case of curves.

By the theorem in section 2(b) it will suffice to prove the following:

Given the situation (2.1), evaluating at a generic point of S we have the Kodaira-Spencer class

$$\sigma \in T^* \otimes H^1(\Sigma_L)$$

associated to the family $\{L_s \to X_s\}$. Using the splitting

$$H^1(\Sigma_L) \cong H^1(\mathcal{O}_X) \oplus H^1(\Theta_X)$$

arising from the assumption $c_1(L) = 0$, we write

$$\sigma = (\tau, \theta) \ ;$$

as was noted in section 2(b), the equivalence class

$$[\tau] \in T^* \otimes H^1(\mathcal{O}_X)/\nabla H^0(\Omega^1_X)$$

is well-defined independently of the above splitting. On the other hand, we have

$$\nu_{\mathcal{L}}(s) = [L_s] \in J(X_s) \ ,$$

and we may locally lift $\nu_{\mathcal{L}}$ to a section $\tilde{\nu}$ of $\mathcal{O}_S \otimes R^1_\pi \mathbb{C}$ (i.e., the bundle over S with fibres $H^1(X_s, \mathbb{C})$). Then

$$\nabla \tilde{\nu} \in R^1_\pi \mathcal{O}_{\mathfrak{X}} / \nabla R^0_\pi \Omega^1_{\mathfrak{X}/S}$$

is well-defined and we may evaluate at a generic point to get

$$[\nabla \tilde{\nu}] \in T^* \otimes H^1(\mathcal{O}_X)/\nabla H^0(\Omega^1_X) \; .$$

We will show that

$$\text{(A.9)} \qquad [\tau] = [\nabla \tilde{\nu}] \; .$$

Proof of (A.9): We follow the notation used in the proof of the theorem in section 2(b). Since $\deg L_s = 0$ we may choose the transition functions $\xi_{\alpha\beta}(s)$ to be constant along X_s. Moreover, we may choose

$$\{\zeta_{\alpha\beta}(s)\} \in H^1(X_s, \mathbb{C})$$

such that

$$\xi_{\alpha\beta}(s) = \exp \zeta_{\alpha\beta}(s) \; .$$

This means we have $\zeta_s \in H^1(X_s, \mathbb{C})$ such that ζ_s maps to L_s under the composite

$$H^1(X_s, \mathbb{C}) \xrightarrow{\exp} H^1(X_s, \mathbb{C}^*) \longrightarrow H^1(X_s, \mathcal{O}^*_X) \; .$$

On the one hand, we have from the proof of the theorem in section 2(b) that

$$[\tau] = \{d\zeta_{\alpha\beta}(s)\} \in T^* \otimes H^1(\mathcal{O}_X)/\nabla H^0(\Omega^1_X) \; .$$

On the other hand, $\{d\zeta_{\alpha\beta}(s)\}$ represents $\nabla\zeta$, and since ζ provides a lifting $\tilde{\nu}$ we are done. □

It remains to show that (cf. [9])

$$\text{(A.10)} \qquad z_s \equiv_{\text{rat}} 0 \;\text{ for general }\; s \Rightarrow \delta \nu_{\mathcal{Z}} = 0 \; .$$

We write

$$z_s = z'_s - z''_s$$

where $z'_s, z''_s \in Y^{(m)}_s$ are effective 0-cycles. We then have correspondingly

$$\mathcal{Z}', \mathcal{Z}'' \subset \mathcal{Y}$$

with

$$\mathcal{Z} = \mathcal{Z}' - \mathcal{Z}'' \ .$$

The assumption in (A.10) has the following implication: First we allow ourselves to shrink S and pass to finite coverings — this means that we may ignore phenomena that occur over a proper subvariety of S, and that any construction that is algebraic in $s \in S$ may be assumed to be rational. Then we may find

$$w_s \in Y_s^{(m)}$$

and

$$f_s : \mathbb{P}^1 \to Y_x^{(m+m)}$$

both varying rationally with s such that

$$\begin{cases} f_s(0) = z_s' + w_s \\ f_s(\infty) = z_s'' + w_s \ . \end{cases}$$

Let γ be a path in $\mathbb{P}^1$ joining 0 to ∞, and denote by $\gamma_s \subset Y_s$ the 1-chain traced out by $f_s(\gamma)$. Then

$$\partial \gamma_s = z_s \ .$$

If we let $\Gamma = \bigcup_{s \in S} \gamma_s$, then remembering that we can ignore phenomena occuring over a proper subvariety of S we will have

(A.11) $$\partial \Gamma = \mathcal{Z} \ .$$

It follows that $[\mathcal{Z}] = 0$, and then as a consequence

$$\delta \nu_{\mathcal{Z}} = 0 \ .$$

Remark: One could formalize the argument by letting $(\mathcal{Y}/S)^{(k)}$ denote the relative k-fold symmetric product of $\mathcal{Y} \to S$, and then constructing a *regular* map

$$F : \mathbb{P}^1 \times S \to (\mathcal{Y}/S)^{(m+n)}$$

such that (with the obvious notation)

$$\begin{cases} F(0 \times S) = \mathcal{Z}' + W \\ F(\infty \times S) = \mathcal{Z}'' + W \ . \end{cases}$$

Then Γ is the image in $\mathcal{Y}$ of $F(\gamma \times S)$ and by construction (A.11) is satisfied.

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