ON A THEOREM OF CHERN

BY

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This paper tells an old story in a more modern setting. One of the more striking results in global differential geometry is the theorem of Chern (Theorem 11 in [1]) relating the characteristic classes of a vector bundle to the curvature of this bundle. A special case of this result is the Gauss-Bonnet theorem for almost complex manifolds.

There are two approaches to characteristic classes which will be discussed here. The first is by using the obstruction theory of fibre bundles as in [1] and [2]. It is here that the differential geometry of the fibre bundle enters essentially, and, although there is a very simple underlying principle, the computations tend to be lengthy and obscure the geometry. Our approach has been first to utilize the geometric principle for simple types of bundles (Proposition 2), where it is hoped that the computations have significant geometric content. Having done this, we pass to more complicated bundles by algebraic means; the geometric interpretation remains the same. The second approach is that introduced by Kodaira and Spencer in applying the theory of Chern classes to algebraic geometry. Again, we try to illustrate in a simple but meaningful way how the curvature enters (Proposition 3) and is useful for the theory of complex manifolds. Again we pass from this result to more complicated bundles by using what might be called the "induction principle in fibre bundle theory" due to Chern [2].

This paper is expository in nature, and the expert will find little that is new. In particular, we have reproduced a proof (Proposition 5) which seems to exist only in lecture notes. The point of view adopted is differential-geometric, and we have freely used results here (reference [4]). On the other hand, we have tried to utilize only general techniques in topology, avoiding specific results. As a single exception, we have used some homology properties of the Grassmann variety in §§4 and 5.

1. Preliminaries

We shall work within the framework of compact almost complex manifolds (abbreviated a.c. manifolds); if X is one such, the a.c. structure tensor is denoted by J_X. An almost complex mapping is one whose differential commutes with the a.c. structure tensors. We define an a.c. principal bundle $GL(r, \mathbf{C}) \to P_\xi \xrightarrow{\pi} X$ (where $GL(r, \mathbf{C}) \to P_\xi$ is injection of the fibre, and π is the bundle projection) such that (i) J_{P_ξ} is invariant under $GL(r,\mathbf{C})$ acting on the right, (ii) J_{P_ξ} restricted to a fibre gives the a.c. structure of the fibre,

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and (iii) $\pi_* J_{P_\xi} = J_X \pi_*$ where π_* is the differential of π (such a mapping is called an a.c. mapping). Relative to a suitable covering $\{\mathfrak{U}_i\}$ of X, P_ξ will be given by transition functions $f_{ij} : \mathfrak{U}_i \cap \mathfrak{U}_j \to GL(r, \mathbf{C})$ where the f_{ij} are a.c. mappings. Associated with P_ξ, there is an a.c. vector bundle $V_\xi \to X$ with fibre $\mathbf{C}^r$; V_ξ is obtained from P_ξ by the usual action of $GL(r, \mathbf{C})$ on $\mathbf{C}^r$. We may find an Hermitian metric $\langle\ ,\ \rangle$ in the fibres of V_ξ; relative to $\{\mathfrak{U}_i\}$, $\langle\ ,\ \rangle$ will be given by functions $h_i : \mathfrak{U}_i \to H(r)$ where $H(r)$ is the manifold of positive definite Hermitian forms on $\mathbf{C}^r$. We write $(h_i)_{\alpha\bar{\beta}}$ for the matrix form of h_i and sometimes denote $\langle\ ,\ \rangle$ by h. The transition rule for h_i in $\mathfrak{U}_i \cap \mathfrak{U}_j$ is

$$h_j = {}^t\bar{f}_{ij} h_i f_{ij}, \tag{1.1}$$

and if $y = (y^1, \cdots, y^r)$, $z = (z_1, \cdots, z_r)$ are sections of V_ξ over $\mathfrak{U}_i$,

$$\langle y, z\rangle = {}^t y h_i \bar{z} = \textstyle\sum_{\alpha,\beta=1}^r (h_i)_{\alpha\bar{\beta}}\, y_\alpha \bar{z}_\beta . \tag{1.2}$$

We now define a connexion in $P_\xi \to X$; it is desirable to have formulae which are useful for calculation. Let $\mathfrak{gl}(r, \mathbf{C})$ be the Lie algebra of $r \times r$ complex matrices with the usual commutation rule: $[A, B] = AB - BA$.

Definition. A *connexion* in $P_\xi \to X$ is a $\mathfrak{gl}(r,\mathbf{C})$-valued form $\omega = (\omega_{\alpha\beta})$ on P_ξ such that

(i) ω restricted to a fibre $\pi^{-1}(x_0) \cong x_0 \times GL(r,\mathbf{C})$ is given by $\omega^\alpha_\beta(x_0, g) = Y^\alpha_\gamma(g) dX^\gamma_\beta(g)$, where $X^\alpha_\gamma(g)$ are the matrix coordinates of g and $(Y^\alpha_\gamma) = (X^\alpha_\gamma)^{-1}$. In shorthand, $\omega \mid \pi^{-1}(x_0) = g^{-1}dg$. (This says that ω gives the Maurer-Cartan form on each fibre.)

(ii) In $\pi^{-1}(\mathfrak{U}) \cong \mathfrak{U} \times GL(r, \mathbf{C})$ with coordinates (u, g),

$$\omega(u, g'g) = g^{-1}\omega(u, g')g = (\operatorname{Ad} g^{-1})\omega(u, g').$$

If we write locally

$$\omega^\alpha_\beta(u, g) = Y^\alpha_\gamma(g)\, dX^\gamma_\beta(g) + \Lambda^\alpha_\beta(u, g),$$

it follows from (ii) that $\Lambda(u, g) = g^{-1}\Lambda(u, e)g$ where $\Lambda(u, e) = \theta(u)$ is a local $\mathfrak{gl}(r,\mathbf{C})$-valued form on $\mathfrak{U} \subset X$. We write

$$\omega(u, g) = g^{-1}\, dg + (\operatorname{Ad} g^{-1})\theta; \tag{1.3}$$

a connexion is given either by ω on P_ξ (global) or by θ on X (local). As usual, ω (or θ) allows us to define the *covariant derivative* D in tensor bundles associated with P_ξ; for example, if $y : \mathfrak{U} \to V_\xi$ is a local section, then $Dy = dy + \theta y$; y is parallel $\Leftrightarrow Dy = 0$.

Proposition 1. *With the Hermitian metric h, we may uniquely associate a connexion θ such that*

(i) *θ is of type* $(1, 0)$ (ω *will be also of type* $(1, 0)$),

(ii) *parallel translation preserves* $\langle\ ,\ \rangle$.

Proof. Let y be a local parallel cross section of V_ξ; then $d\langle y, y\rangle = 0$. By (1.2),

$$0 = d({}^t\bar{y}hy) = {}^td\bar{y}hy + {}^t\bar{y}dhy + {}^t\bar{y}hdy,$$

and since $dy = -\theta y$, $d\bar{y} = -\bar{\theta}\bar{y}$, we have

$${}^t\bar{y}(-{}^t\bar{\theta}h + dh - h\theta)y = 0,$$

and hence, by (i)

$$\partial h = h\theta, \qquad \bar{\partial} h = {}^t\bar{\theta}h$$

(∂ and $\bar{\partial}$ are the type components of d). By setting $\theta = h^{-1}\partial h$,

$${}^t\bar{\theta} = {}^t\bar{\partial}\bar{h}\,{}^t\bar{h}^{-1} = \bar{\partial}h\,h^{-1} \qquad (\text{since } {}^t\bar{h} = h),$$

and we are done.

We let $\mathrm{End}(V_\xi)$ with fibre $\mathfrak{gl}(r, \mathbf{C})$ be the bundle of endomorphisms of V_ξ. If Λ, Ξ are $\mathrm{End}(V_\xi)$-valued forms, we may define $\Lambda \wedge \Xi$ by roofing entries; if Λ, Ξ are of degrees p, q respectively, we define

$$[\Lambda, \Xi] = \Lambda \wedge \Xi + (-1)^{pq+1}\Xi \wedge \Lambda.$$

The *curvature form* on X of θ is given by

(1.4) $$\Theta = d\theta + \theta \wedge \theta \quad (\textit{Cartan structure equation});$$

the curvature Ω of ω on P_ξ is given by

(1.4)′ $$\Omega = D\omega = d\omega + \omega \wedge \omega.$$

Using (1.3), one checks easily that in $\pi^{-1}(\mathfrak{U}) \times GL(r,\mathbf{C})$, $\Omega(u, g) = \mathrm{Ad}\, g^{-1}\Theta(u, e)$, which shows that Θ is a global $\mathrm{End}(V_\xi)$-valued 2-form on X. Thus in $\mathfrak{U}_i \cap \mathfrak{U}_j$,

(1.5) $$\Theta_i = (\mathrm{Ad}\, f_{ij})\Theta_j.$$

Now if $\theta = h^{-1}\partial h$ is a metric connexion, then

$$\begin{aligned}\Theta &= d(h^{-1}\partial h) + h^{-1}\partial h \wedge h^{-1}\partial h\\ &= \partial(h^{-1}\partial h) + h^{-1}\partial h \wedge h^{-1}\partial h + \bar{\partial}(h^{-1}\partial h),\end{aligned}$$

and since $\partial h^{-1} = -h^{-1}\partial h h^{-1}$,

(1.6) $$\Theta = \bar{\partial}\theta = \bar{\partial}(h^{-1}\partial h).$$

This section will be terminated by discussing a few classical identities. By definition,

$$\begin{aligned}D\Theta &= d\Theta + [\theta, \Theta] = d\Theta + \theta \wedge \Theta - \Theta \wedge \theta\\ &= d\theta \wedge \theta - \theta \wedge d\theta + \theta \wedge d\theta + \theta \wedge \theta \wedge \theta - d\theta \wedge \theta - \theta \wedge \theta \wedge \theta;\end{aligned}$$

i.e.,

(1.7) $$D\Theta = 0 \qquad (\textit{Bianchi identity}).$$

If Λ is an $\mathrm{End}(V_\xi)$-valued 1-form, then

$$D\Lambda = d\Lambda + [\theta, \Lambda] = d\Lambda + \theta \wedge \Lambda + \Lambda \wedge \theta, \tag{1.8}$$

and by the same calculation as above (1.7),

$$D^2\Lambda = \Theta \wedge \Lambda - \Lambda \wedge \Theta = [\Theta, \Lambda] \qquad (\textit{Ricci identity}). \tag{1.9}$$

2. Statement of the theorem

Let P be a symmetric multilinear form on $\mathfrak{gl}(r, \mathbf{C})$ which is Ad-invariant; i.e., P is a polynomial on $\mathfrak{gl}(r, \mathbf{C})$ such that

$$P(\mathrm{Ad}\, gA_1, \cdots, \mathrm{Ad}\, gA_s) = P(A_1, \cdots, A_s) \qquad (g \in GL(r, \mathbf{C}), A_j \in \mathfrak{gl}(r, \mathbf{C})). \tag{2.1}$$

Now if $B \in \mathfrak{gl}(r, \mathbf{C})$, e^{tB} is a one-parameter subgroup of $GL(r, \mathbf{C})$, and

$$\frac{d}{dt}(\mathrm{Ad}\, e^{tB})(A)\Big]_{t=0} = \frac{d}{dt}(e^{tB}Ae^{-tB})\Big]_{t=0} = BA - AB.$$

Thus in (2.1), if we let g vary along a one-parameter subgroup and differentiate at $t = 0$, we have

$$\textstyle\sum_i P(A_1, \cdots, [B, A_i], \cdots, A_s) = 0. \tag{2.2}$$

The set of all such P's forms a ring (usual multiplication of symmetric forms) which we denote by I. Given a curvature form Θ and $P \in I$, we assert that the expression $P(\Theta) = P(\Theta, \cdots, \Theta)$ makes sense and is a closed global scalar form on X. First, since Θ is an $\mathrm{End}(V_\xi)$-valued 2-form, $P(\Theta_i)$ makes sense in $\mathfrak{U}_i$ and therefore in $\mathfrak{U}_i \cap \mathfrak{U}_j$ by (1.5) and (2.1); secondly, by (1.7),

$$dP(\Theta) = \textstyle\sum P(\Theta, \cdots, D\Theta, \cdots, \Theta) = 0.$$

We may thus define a homomorphism $W : I \rightarrow H^*(X, C)$ called the *Weil homomorphism* by $W(P) = P(\Theta)$.

For $0 \leqq j \leqq r$, we define $P_j(B) = P_j(\overbrace{B, \cdots, B}^{j})$ by

$$\det(\lambda I + B) = \textstyle\sum_{j=0}^{r} (-1)^j P_j(B)\lambda^{r-j} \qquad (B \in \mathfrak{gl}(r, \mathbf{C})). \tag{2.3}$$

The theorem of Chern is the following.

THEOREM. *Let $V_\xi \rightarrow X$ be an a.c. bundle as defined in* §1, *and let Θ be a curvature as constructed in* §1. *Then, if $c_j(V_\xi)$ is the j^{th} characteristic class of V_ξ and if $\Xi = (2\pi i)^{-1}\Theta$, under the de Rham isomorphism*

$$c_j(V_\xi) = P_j(\Xi), \tag{2.4}$$

where P_j is given by (2.3).

There are several extant definitions of $c_j(V_\xi)$; we shall show that (2.4) is valid where c_j is defined by obstruction theory [1] or by axioms [3].

3. Line bundles

A *line bundle* is by definition an a.c. bundle $V_\xi \to X$ where $r = 1$; the principal bundle P_ξ has fibre $\mathbf{C}^* = \mathbf{C} - 0$. An Hermitian metric $\langle\ ,\ \rangle$ is given (relative to $\{\mathfrak{U}_i\}$) by positive real functions a_i defined in $\mathfrak{U}_i$ such that

$$a_i\,|f_{ij}|^2 a_j^{-1} = 1 \qquad \text{(see (1.1))}. \tag{3.1}$$

If $\mathfrak{U} \subset X$ is an open set such that $\pi^{-1}(\mathfrak{U})$ is a product $\mathfrak{U} \times C^*$ with coordinates (u, z), then

$$\omega_{\mathfrak{U}} = z^{-1}dz + z^{-1}\theta_{\mathfrak{U}}\, z = d\log z + \theta_{\mathfrak{U}}\,. \tag{3.2}$$

Now

$$D\omega_{\mathfrak{U}} = d\omega_{\mathfrak{U}} + \omega_{\mathfrak{U}} \wedge \omega_{\mathfrak{U}} = \Omega_{\mathfrak{U}} = \operatorname{Ad} z^{-1}(\Theta_{\mathfrak{U}}) \qquad ((1.4), (1.4)'),$$

and since $\omega_{\mathfrak{U}} \wedge \omega_{\mathfrak{U}} = 0$,

$$d\omega = \pi^*(\Theta). \tag{3.3}$$

The geometry of Chern classes as obstruction classes is essentially given by the following proposition.

PROPOSITION 2. *The first Chern class of the principal line bundle in the sense of obstruction theory is represented under the de Rham isomorphism by* $-(2\pi i)^{-1}\Theta$.

Proof. We first recall the obstruction definition. Let X be triangulated as a finite complex K, and denote the i-skeleton by K^i. Subdividing if necessary, we may suppose, for each 2-simplex s^2, that $\pi^{-1}(s^2)$ is a product $s^2 \times \mathbf{C}^*$. Letting $\pi_k = k^{\text{th}}$ homotopy group, since $\pi_0(\mathbf{C}^*) = 0$, we may get a smooth cross section $\rho : K^1 \to P_\xi$ over the 1-skeleton. Thus if s^2 is a 2-simplex, we have defined a mapping $\rho : \partial(s^2) \to P_\xi \mid \partial(s^2)$ (∂ = boundary), and since $\pi^{-1}(s^2)$ is a product, we have a mapping $\rho : \partial(s^2) \to \mathbf{C}^*$. This mapping defines an element of $\pi_1(C^*) \cong \mathbf{Z}$, and since P_ξ is an oriented fibre bundle, we have assigned to each s^2 an integer, the degree of $\rho : \partial(s^2) \to \mathbf{C}^*$. The obstruction cocycle $-f$ assigns to each s^2 this integer; as a cocycle, $-f$ is independent of ρ.

Let $s_1^2, \cdots, s_m^2$ be the 2-simplices of K^2; we must show that if $\sigma^2 = \sum c_i s_i^2$ and $\partial(\sigma^2) = 0$, then

$$\sum c_i f(s_i^2) = (2\pi i)^{-1} \int_{\sigma^2} \Theta. \tag{3.4}$$

We set $\delta_j = f(s_j^2)$, and we may assume that for some k, $\delta_j = 0$ for $j > k$. For $j > k$, ρ may be extended to s_j^2, and for $1 \leqq j \leqq k$, ρ may be defined on s_j^2 with the exception of a single point $p_j \in s_j^2$; let D_j be a small disc around p_j. The following notations will be used in the calculation below: the symbol "$\sim$" means "approximately equal to", $\Gamma = \{z : |z| = 1\}$, and * is the induced mapping on forms.

$$\begin{aligned}
\int_{\sigma^2}\Theta &= \sum c_j \int_{s_j^2}\Theta \\
&\sim \sum_{1\leqq j\leqq k} c_j \int_{s_j^2-D_j}\Theta + \sum_{k<l\leqq m} c_l \int_{s_l^2}\Theta \\
&= \sum_{1\leqq j\leqq k} c_j \int_{s_j^2-D_j}\rho^*\pi^*\Theta + \sum_{k<l\leqq m} c_l \int_{s_l^2}\rho^*\pi^*\Theta \\
&= \sum_{1\leqq j\leqq k} c_j \int_{s_j^2-D_j}\rho^*\Omega + \sum_{k<l\leqq m} c_l \int_{s_l^2}\rho^*\Omega \\
&= \sum_{1\leqq j\leqq k} c_j \int_{s_j^2-D_j} d\rho^*\omega + \sum_{k<l\leqq m} c_l \int_{s_l^2} d\rho^*\omega \qquad ((3.3)) \\
&= \sum_{1\leqq j\leqq k} c_j \int_{\partial(s_j^2-D_j)}\rho^*\omega + \sum_{k<l\leqq m} c_l \int_{\partial(s_l^2)}\rho^*\omega \\
&= \sum_{1\leqq j\leqq m} c_j \int_{\partial(s_j^2)}\rho^*\omega - \sum_{1<l\leqq k} c_l \int_{\partial(D_l)}\rho^*\omega \\
&= -\sum_{1\leqq l\leqq k} c_l \int_{\rho(\partial(D_l))}\omega \qquad (\text{since } \partial(\sigma^2) = 0).
\end{aligned}$$

Now by (3.2), the definition of δ_l, and since D_l is negatively oriented,

$$\begin{aligned}
\int_{\sigma^2}\Theta &\sim + \sum_{1\leqq l\leqq k} c_l \int_{\delta_l\Gamma} d\log z + \varepsilon\theta \qquad (\varepsilon \text{ small}) \\
&\sim \sum_{1\leqq l\leqq k} (2\pi i)\delta_l c_l .
\end{aligned}$$

By shrinking D_l to p_l, the $\sim$ becomes $=$, and (3.4) is proved.

In [3, page 61], the Chern classes were given axiomatically; we shall see that, if $c_j'(V_\xi)$ denotes the j^{th} Chern class defined axiomatically,

$$c_j'(V_\xi) = P_j(\Xi). \tag{3.5}$$

For the moment we work on line bundles; it is assumed that X is a complex manifold (J_X is integrable) and that $P_\xi \xrightarrow{\pi} X$ is a holomorphic line bundle. Over X there is the exact sheaf sequence

$$0 \to \mathbf{Z} \to \mathcal{C} \xrightarrow{\exp} \mathcal{C}^* \to 0$$

[3, page 27] where e.g., $\mathcal{C}$ is the sheaf of holomorphic functions on X. The holomorphic line bundles are given by $H^1(X, \mathcal{C}^*)$ [3, page 42], and in the exact cohomology sequence

$$\cdots \to H^1(X, \mathcal{C}^*) \xrightarrow{\delta} H^2(X, \mathbf{Z}) \to \cdots ,$$

if $\xi \in H^1(X, \mathcal{C}^*)$ gives P_ξ, then $c_1'(V_\xi) = c_1'(\xi) = \delta(\xi)$ according to [3, Satz 4.3.1].

PROPOSITION 3. *If $\xi \in H^1(X, \mathcal{C}^*)$ is provided with a metric curvature Θ, then under the de Rham isomorphism*

$$-(2\pi i)^{-1}\Theta = \delta(\xi). \tag{3.6}$$

Proof. Let $\{\mathfrak{U}_i\}$ be a sufficiently fine covering relative to which ξ is given by a holomorphic cocycle $\xi_{ij} : \mathfrak{U}_i \cap \mathfrak{U}_j \to \mathbf{C}^*$. Then, by the definition of δ, $\delta(\xi)$ is given by the integral cocycle c_{ijk} where

$$c_{ijk} = (2\pi i)^{-1}(\log \xi_{ij} - \log \xi_{ik} + \log \xi_{jk}). \tag{3.7}$$

Letting $N(\mathfrak{U}) =$ nerve of $\{\mathfrak{U}_i\}$, we shall trace the Čech cocycle

$$c_{ijk} \in Z^2(N(\mathfrak{U}), \mathbf{Z}) \subset Z^2(N(\mathfrak{U}), \mathbf{C})$$

through the de Rham isomorphism as given in [3, pp. 38–40] and find that it is represented by $-(2\pi i)^{-1}\Theta$.

(a) The de Rham isomorphism goes as follows: given a cocycle

$$\gamma = \{\gamma_{ijk}\} \in Z^2(N(\mathfrak{U}), \mathbf{C}),$$

there exist C^∞ functions σ_{ij} in $\mathfrak{U}_i \cap \mathfrak{U}_j$ such that

$$\sigma_{ij} - \sigma_{ik} + \sigma_{jk} = \gamma_{ijk} \qquad (\text{in } \mathfrak{U}_i \cap \mathfrak{U}_j \cap \mathfrak{U}_k).$$

Then $d\sigma_{ij} + d\sigma_{jk} = d\sigma_{ik}$, and there exist 1-forms τ_i in $\mathfrak{U}_i$ such that

$$\tau_i - \tau_j = d\sigma_{ij} \qquad (\text{in } \mathfrak{U}_i \cap \mathfrak{U}_j).$$

However, in $\mathfrak{U}_i \cap \mathfrak{U}_j$, $d\tau_i = d\tau_j$, and thus there is a global closed 2-form τ such that $\tau \mid \mathfrak{U}_i = d\tau_i$; τ is the de Rham class corresponding to γ.

(b) Let ξ_{ij}, c_{ijk} be as above, and let a_i be an Hermitian metric in P_ξ. By (3.7), we may choose the σ_{ij} of (a) to be $(2\pi i)^{-1} \log \xi_{ij}$, and then $d\sigma_{ij} = (2\pi i)^{-1}\partial \log \xi_{ij}$ (since ξ_{ij} is holomorphic). By (3.1),

$$\log a_i + \log \xi_{ij} + \log \bar{\xi}_{ij} - \log a_j = 0$$

and thus

$$\partial \log a_i - \partial \log a_j = -\partial \log \xi_{ij} \qquad (\text{in } \mathfrak{U}_i \cap \mathfrak{U}_j).$$

Thus we may take $\tau_i = -(2\pi i)^{-1}\partial \log a_i$ and

$$\tau = \{d\tau_i\} = \{-(2\pi i)^{-1}\bar{\partial}\partial \log a_i\} = -(2\pi i)^{-1}\bar{\partial}\partial \log a_i .$$

Taking into account (1.6), we are done.

4. Sum of line bundles

If V_ξ is an r-dimensional a.c. vector bundle, one may define the Chern classes $c_0(V_\xi) = 1, c_1(V_\xi), \cdots, c_r(V_\xi)$ by obstruction theory as in the case $r = 1$. In particular, if $V_\xi = V_{\xi_1} \oplus \cdots \oplus V_{\xi_r}$ is a Whitney sum of r line bundles, we would like to write $c_j(V_\xi)$ in terms of the $c_1(V_{\xi_k})$. The answer is given by the duality theorem (Theorem 7 in [1]).

THEOREM. *If* $V_\xi = V_{\xi_1} \oplus \cdots \oplus V_{\xi_r}$, *we write*

$$c(V_\xi) = 1 + c_1(V_\xi) + \cdots + c_r(V_\xi),$$

$$c(V_{\xi_k}) = 1 + c_1(V_{\xi_k}) \qquad (k = 1, \cdots, r).$$

Then we have

$$c(V_\xi) = c(V_{\xi_1}) \cdot \cdots \cdot c(V_{\xi_r}). \tag{4.1}$$

The formula (4.1) is taken as an axiom in [3] and, as mentioned above, is given an obstruction-theoretic proof in [1]. The fact that the forms given in (2.4) satisfy (4.1) is made clear by the following remark: If $V_\xi = V_{\xi_1} \oplus \cdots \oplus V_{\xi_r}$, then $c_j(V_\xi) = \sigma_j(c_1(V_{\xi_1}), \cdots, c_1(V_{\xi_r}))$ where σ_j is the j^{th} elementary symmetric function of its variables. Thus, in view of Proposition 2, we may state

PROPOSITION 4. *Let* $V_\xi = V_{\xi_1} \oplus \cdots \oplus V_{\xi_r}$; *then* (2.4) *holds for* V_ξ *where* $c_j(V_\xi)$ *is defined either by the axioms or by obstruction theory. In fact,*

$$c_j(V_\xi) = (-1)^j \sigma_j(\Xi_1, \cdots, \Xi_r), \tag{4.2}$$

where $\Xi_k = (2\pi i)^{-1}\Theta_k$ *and* Θ *is a curvature in* P_{ξ_k}.

5. Completion of the proof

There are two more steps in the proof: first we discuss the contravariant nature of I under mappings; then we use a mapping to split a vector bundle into a sum of line bundles and use §4.

Let $V_\xi \xrightarrow{\pi} X$ be an a.c. vector bundle, and let $\sigma : X' \to X$ be an a.c. mapping. Then σ induces an a.c. vector bundle $\sigma^*(V_\xi)$ over X' as follows: $\sigma^*(V_\xi) \subset X' \times V_\xi$ consists of those points (x', v) such that $\sigma(x') = \pi(v)$; $\pi' : \sigma^*(V_\xi) \to X'$ is defined by $\pi'(x', v) = x'$. If $\{f_{ij}\}$ are the transition functions of V_ξ relative to $\{\mathfrak{U}_i\}$, then $\{f_{ij} \circ \sigma\}$ are the transition functions of $\sigma^*(V_\xi)$ relative to $\{\sigma^{-1}(\mathfrak{U}_i)\}$. Let $\langle\ ,\ \rangle$ be an Hermitian metric in V_ξ; for $v, w \in \pi^{-1}(x)$, we have defined $\langle v, w\rangle$. There is an induced metric $\langle\ ,\ \rangle'$ in $\sigma^*(V_\xi)$ where if $(x', v)(x', w) \in (\pi')^{-1}(x')$, $\langle (x', v), (x', w)\rangle = \langle v, w\rangle$. In $\sigma^{-1}(\mathfrak{U}_i)$, the Hermitian metric h_i' is given by $h_i' = h_i \circ \sigma$. The following equations are easily checked:

$$\text{(i)} \qquad \sigma^*\theta = \theta' \qquad (\theta, \theta' \text{ defined as in §1}),$$

$$(5.1) \quad \text{(ii)} \qquad \sigma^*\Theta = \Theta',$$

$$\text{(iii)} \qquad \sigma^*P(\Theta) = P(\Theta') \qquad (P \in I).$$

For example, $\theta' = (h')^{-1}\partial(h') = (h \circ \sigma)^{-1}\partial(h \circ \sigma) = \sigma^*(h^{-1}\partial h) = \sigma^*\theta$.

Now we discuss what it "means" to split a vector bundle. The general principle is this: Let G be a Lie group, $H \subset G$ a closed subgroup, and suppose that $G \to P_\xi \to X$ is a principal bundle. Then G acts on the right on P_ξ as does H, and we have a diagram

$$\begin{array}{ccc} P_\xi & \xrightarrow{\quad H \quad} & P_\xi/H \\ & {\scriptstyle G}\searrow \quad \swarrow{\scriptstyle G/H} & \\ & X & \end{array}$$

where every map is a fibering map and the appropriate fibres are written in. The bundle $\sigma^*(P_\xi) \to P_\xi/H$ is a priori a principal bundle with group G; however the structure group in this case may always be reduced to H. If G, H are complex Lie groups, P_ξ is a.c., then this reduction will be an a.c. reduction.

Returning to the situation of §1, we let $\Delta \subset GL(r, \mathbf{C})$ be the subgroup consisting of matrices of the form

$$\begin{pmatrix} * & \cdot & \cdot & \cdot & * \\ 0 & * & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & * \end{pmatrix},$$

apply the above reduction where $G = GL(r, \mathbf{C})$, $H = \Delta$, and conclude that $\sigma^*(P_\xi) \to P_\xi/\Delta$ is an a.c. bundle with group Δ and therewith splits *topologically*. More specifically, let $\langle\ ,\ \rangle^{\#}$ be an Hermitian metric in

$$\Delta \to \sigma^*(P_\xi) \to P_\xi/\Delta$$

given by an Hermitian form $h^{\#}$. We may suppose that $h^{\#}$ gives a reduction of the structure of $\sigma^*(V_\xi) \to P_\xi/\Delta$ to $T \subset \Delta$, where T consists of matrices of the form

$$\begin{pmatrix} e^{i\theta_1} \cdots 0 \\ \vdots \qquad \vdots \\ 0 \quad \cdots e^{i\theta_r} \end{pmatrix};$$

this is a splitting into a sum of line bundles.

Now if h is a metric in $V_\xi \to X$, then we have a metric $h' = h \circ \sigma$ in $\sigma^*(V_\xi) \to P_\xi/\Delta$ as constructed above. Thus for $P \in I$, we have two distinguished elements in $H^*(P_\xi/\Delta, \mathbf{C})$; namely $P(\Theta^{\#})$ and $P(\Theta') = \sigma^* P(\Theta)$. The theorem will be proved if we can show the following two propositions:

PROPOSITION 5. *As elements of* $H^*(P_\xi/\Delta, \mathbf{C})$,

$$P(\Theta^{\#}) \sim P(\Theta'). \tag{5.2}$$

PROPOSITION 6. *In the diagram*

$$\begin{array}{ccc} P_\xi & \longrightarrow & P_\xi/\Delta \\ & \searrow \quad \swarrow \sigma & \\ & X & \end{array}$$

the cohomology mapping $\sigma^* : H^*(X, \mathbf{C}) \to H^*(P_\xi/\Delta, \mathbf{C})$ *is injective.*

These results are due to Weil and Borel respectively.

Proof of Proposition 6. For any manifold Y, we denote $H^*(Y, \mathbf{C})$ by $H^*(Y)$; it will suffice to show that $H^*(P_\xi/\Delta)$ has the same additive structure as $H^*(G/\Delta) \otimes H^*(X)$. By working in the classifying space, and since G/Δ is differentiably isomorphic to $\mathfrak{U}(r)/T^r = F(r)$, we may show that, in the fibering

$$(5.3)\qquad F(r) \to \mathfrak{U}(r+N)/T^r \times \mathfrak{U}(N) \xrightarrow{\sigma} H(r, N)$$
$$(\mathfrak{U}(r)/T^r \to \mathfrak{U}(r+N)/T^r \times \mathfrak{U}(N) \to \mathfrak{U}(r+N)/\mathfrak{U}(r) \times \mathfrak{U}(N)),$$

σ^* is injective. This is done in two steps.

(a) All elements in $H^*(F(r))$ are of even degree; more precisely, there exist elements $x_1, \cdots, x_{r-1} \in H^2(F(r))$ such that the elements $x_1^{a_1}, \cdots, x_{r-1}^{a_{r-1}}$ $(0 \leqq a_i \leqq r - i)$ form an additive basis for $F(r)$.

Proof. We proceed by induction using the spectral sequence of the fibering $F(r-1) \to F(r) \to P_{r-1}(\mathbf{C})$. By the induction hypothesis, $E_2^{p,q} = 0$ unless $p = 2a$, $q = 2b$, and $E_2^{2a,2b} \cong H^{2b}(P_{r-1}(\mathbf{C})) \otimes H^{2a}(F(r-1))$. If we take a basis of $H^*(F(r-1))$ of the form $x_2^{a_2}, \cdots, x_{r-1}^{a_{r-1}}$ $(0 \leqq a_j \leqq r - j - 1)$ and let x_1 be the generator of $H^*(P_{r-1}(\mathbf{C}))$, the conclusion follows since all differentials in the spectral sequence are zero.

(b) In the fibering (5.3), all cohomology of the base and fibre is in even dimensions, and (by a spectral sequence again) $H^*(\mathfrak{U}(r+N)/T \times \mathfrak{U}(N))$ is additively isomorphic to $H^*(F(r)) \otimes H^*(H(r, N))$.

Proof of Proposition 5 (Weil). Consider the situation of §1, and let θ, $\tilde{\theta}$ be two connexions in $P_\xi \to X$; then $\eta = \theta - \tilde{\theta}$ is a global $\mathrm{End}(V_\xi)$-valued form. By (1.4),

$$\tilde{\Theta} = d\tilde{\theta} + \tilde{\theta} \wedge \tilde{\theta} = d\theta - d\eta + (\theta - \eta) \wedge (\theta - \eta)$$
$$= \Theta - d\eta - [\theta, \eta] + \eta \wedge \eta,$$

and by (1.8),

$$(5.4)\qquad \tilde{\Theta} - \Theta = -D\eta + \eta \wedge \eta.$$

We set

$$P(A) = P(\overbrace{A, \cdots, A}^{s}) \quad \text{and} \quad Q(A, B) = (A, B, \cdots, B)$$

for $A, B \in \mathfrak{gl}(r, \mathbf{C})$. Define $F(t) = P(A - tB - t^2C) - P(A)$, so that

$$P(A - B - C) - P(A) = \int_0^1 F'(t)\,dt,$$

where, since P is symmetric, $F'(t) = -sQ(B + 2tC, A - tB - t^2C)$. Now take $A = \Theta$, $B = D\eta$, $C = -\eta \wedge \eta$ so that by (5.4), we have

$$(5.5)\quad P(\tilde{\Theta}) - P(\Theta) = -s\int_0^1 Q(D\eta - 2t\eta \wedge \eta, \Theta - tD\eta + t^2\eta \wedge \eta)\,dt.$$

On the other hand, by (1.8) and (1.9),

$$\begin{aligned} dQ(\eta, \Theta - tD\eta + t^2\eta \wedge \eta) &= Q(D\eta, \Theta - tD\eta + t^2\eta \wedge \eta) \\ &\quad + (s-1)P(\eta, -t[\Theta, \eta] + t^2[D\eta, \eta], \Theta - tD\eta + t^2\eta \wedge \eta, \cdots) \\ &= Q(D\eta, \Theta - tD\eta + t^2\eta \wedge \eta) \\ &\qquad + (s-1)P(\eta, t[\eta, -tD\eta + \Theta], \Theta - tD\eta + t^2\eta \wedge \eta, \cdots). \end{aligned}$$

By (2.2), and since $[\eta, \eta \wedge \eta] = 0$,

$$\begin{aligned} dQ(\eta, \Theta - tD\eta + t^2\eta \wedge \eta) &= Q(D\eta, \Theta - tD\eta + t^2\eta \wedge \eta) \\ &\quad - Q(2t\eta \wedge \eta, \Theta - tD\eta + t^2\eta \wedge \eta) \\ &= Q(D\eta - 2t\eta \wedge \eta, \Theta - tD\eta + t^2\eta \wedge \eta). \end{aligned}$$

Then by (5.5),

$$P(\tilde{\Theta}) - P(\Theta) = d\left(-s\int_0^1 Q(\eta, \Theta - tD\eta + t^2\eta \wedge \eta)\,dt\right), \quad \text{Q.E.D.}$$

6. Concluding remarks

(i) The Chern classes $c_j'(V_\xi)$ were defined axiomatically in [3]; by setting $\tilde{c}_j(V_\xi) = P_j(\Xi)$, the classes $\tilde{c}_j$ verify the axioms. Axiom I is trivial, Axiom II follows from §5, and Axiom III from §4. Finally, Axiom IV follows from Proposition 3 by taking $X = P_n(\mathbf{C})$ and P_ξ to be the line bundle of a hyperplane section (whence the $(-1)^j$ sign in defining P_j).

(ii) In §2, the obstruction class of an a.c. line bundle $P_\xi \xrightarrow{\pi} X$ was discussed. The proof of Proposition 2 shows that the following is true:

PROPOSITION. *The form $\Xi = (2\pi i)^{-1}\Theta$ representing $c_1(V_\xi)$ has the following property: there exists a global form ϖ $(= (2\pi i)^{-1}\omega)$ on P_ξ such that $d\varpi = \pi^*(\Xi)$ and $\varpi \mid$ fibre gives the fundamental class of the fibre. Briefly stated, in the fibering $\mathbf{C}^* \to P_\xi \to X$, $c_1(V_\xi)$ is the transgression of the generator of $H^1(\mathbf{C}^*, \mathbf{Z})$.*

This statement (suitably interpreted) is true for $r > 1$ and $c_j(V_\xi)$; this is Theorem 8 in [1]. The proof follows from the above proposition coupled with the fact that transgression behaves under direct sum and mappings in the same way as the Chern classes.

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