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The Regulator Map for a General Curve

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1. Introduction

For X a smooth projective curve over $\mathbf{C}$, a reasonably mysterious object is the **$\mathbf{K_2}$-group of** X, $K_2(X)$. The piece of this we want to study is

$$\bar{K}_2(X) = Gr_\gamma^2 K_2(X) = H^0(\mathcal{K}_2(\mathcal{O}_X)),$$

where γ is the **weight filtration by Adams operations**. This has a beautiful transcendental invariant attached to it, the **regulator map**

$$R_X : \bar{K}_2(X) \to H^1(X, \mathbf{C}/\mathbf{Z}(2)).$$

If p_0 is a base-point on X, for a loop γ on X, the regulator is defined by

$$R_X(\{f,g\})(\gamma) = \int_\gamma \log(f)\frac{dg}{g} - \log(g(p_0))\frac{df}{f},$$

which is well-defined mod $(2\pi i)^2\mathbf{Z} = \mathbf{Z}(2)$ independent of p_0 and of choices of branches of log, and depends only on the homology class of γ. Out of this map, an arithmetic version can be made if X is defined over a number field k by using the appropriate parts of R_X for the various complex embeddings of k, cf [R]. The image of the arithmetic regulator is not at all well understood, and one has important conjectures about it when X is defined over a number field, due to Bloch-Beilinson, described for example in [R].

The image of R_X is likewise highly mysterious. A basic result, a proof of which we will give below, is [R]:

THEOREM 1.1. *(Beilinson) For a fixed X, R_X is constant on algebraic families.*

Thus the image of R_X is 0-dimensional in all cases. We adopt the language that **very general** means that a statement holds on the complement of a countable number of lower-dimensional Zariski closed subsets; **general** means the statement holds on the complement of a finite number of lower-dimensional Zariski closed subsets. Our results are:

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THEOREM 1.2.
(1) For any smooth projective complex curve X, the image of R_X contains the torsion points of $H^1(X, \mathbf{C}/\mathbf{Z}(2))$.
(2) For a very general complex curve X of genus ≥ 2, the image of R_X is exactly the torsion points of $H^1(X, \mathbf{C}/\mathbf{Z}(2))$.
(3) Let k be a number field. For a given genus $g \geq 2$, there does not exist a degree d and a number N such that a general curve X defined over k of genus g has an element of $H^0(\mathcal{K}_2(\mathcal{O}_{X(k)}))$ whose regulator is non-torsion and which is a product of $\leq N$ symbols of functions of degree $\leq d$.

REMARKS:
(a) Another way of saying (1) and (2) is that the image of R_X for a very general curve of genus ≥ 2 is

$$H^1(X, \mathbf{Q}(2)/\mathbf{Z}(2)) \subset H^1(X, \mathbf{C}/\mathbf{Z}(2)).$$

As will be discussed later on, if Beilinson's conjectures are correct, this implies: For the very general curve X of genus ≥ 2, in the conjectural category of mixed motives over $\mathbf{C}$,

$$\mathrm{Ext}^1_{\mathcal{MM}_\mathbf{C}}(\mathbf{Q}, H^1(X)(2)) = 0$$

and presumably

$$\mathrm{Ext}^1_{\mathcal{MM}_\mathbf{C}}(\mathbf{Z}, H^1(X)(2)) \cong H^1(X, \mathbf{Q}(2)/\mathbf{Z}(2)).$$

(b) Statement (3) is an elementary consequence of (2), since the existence of elements of $H^0(\mathcal{K}_2(\mathcal{O}_X))$ expressed as the product of N symbols involving functions of given degree defines a variety over $\mathbf{Q}$ in moduli, and by (2) this subvariety must be proper if the regulator is not identically torsion. Also, it is elementary by the Riemann-Roch Theorem that over $\mathbf{C}$ any element of $\mathbf{C}(X)$ can be written as a product of symbols involving only functions of degree $\leq g+2$. Statement (3) says that if the Bloch-Beilinson Conjecture about the image of the regulator of curves over number fields (see [R]), which predicts the existence of elements of $H^0(\mathcal{K}_2(\mathcal{O}_{X(k)}))$ mapping to non-torsion elements under R_X is true, then for a given genus $g \geq 2$ and given algebraic number field k, the number of symbols involving functions of degree $\leq g+2$ required to express any such element is unbounded.

(c) Quite a bit is known about the torsion in $H^0(\mathcal{K}_2(\mathcal{O}_X))$, see [T], [S].

In [C], Collino shows that the image of the regulator map for a very general plane curve of degree ≥ 4 is contained in the torsion points. Interestingly, neither his result nor our result implies the other. The proofs are similar, but rely on different algebraic lemmas to show the vanishing of the infinitesimal invariant. In the same paper, Collino also shows that for genus 1, the image of the regulator map is not finitely generated.

The following more geometric conjecture can be reduced to the general conjectures of Beilinson:

CONJECTURE 1.1. *Modulo torsion, the kernel of R_X is $K_2(\mathbf{C})$.*

From the conjecture and our theorem, it would follow that:
Consequence of the Conjecture. For a general curve X of genus ≥ 2,

$$\frac{\bar{K}_2(X)}{K_2(\mathbf{C})} \text{ is torsion.}$$

REMARKS:
(1) The classes in $\bar{K}_2(X)$ which Bloch-Beilinson predict should exist for $g \geq 1$ when X is defined over $\bar{\mathbf{Q}}$ cannot, for $g \geq 2$, come from some "universal elements" which exist on the general curve.
(2) There are two ways in which the situation for the image of the regulator differs from that of the image of the Abel-Jacobi map; firstly, there is no "abelian part" of $H^1(X, \mathbf{C}/\mathbf{Z}(2))$ and no continuous families, and secondly the fact that all torsion classes appear in the image of the regulator is quite surprising and, at least by us, totally unanticipated–for example, one expects that the situation is quite different for algebraic cycles.

2. Motivic Interpretation of the Regulator Map

In this section, we want to show how the regulator map and various conjectures about it fit into the general framework of the conjectures of Bloch-Beilinson. Indeed, we feel that this is one case where one can "hold in one's hand" the relevant objects and get a reasonable geometric feeling for the content of the conjectures. In what follows, X is a smooth projective curve over $\mathbf{C}$.

PROPOSITION 2.1. *The following are three equivalent definitions of $\bar{K}_2(X)$.*
(1) $\bar{K}_2(X) = H^0(\mathcal{K}_2(\mathcal{O}_X))$;
(2) $\bar{K}_2(X) = \ker(T\colon K_2(\mathbf{C}(X)) \to \oplus_{x\in X}\mathbf{C}^*_x)$, where T is the **tame symbol***;*
(3) $\bar{K}_2(X) \otimes_\mathbf{Z} \mathbf{Q} = CH^2(X,2) \otimes_\mathbf{Z} \mathbf{Q}$.

PROOF: The equivalence of (1) and (2) follows from the **Gersten-Quillen exact sequence**

$$0 \to \mathcal{K}_2(\mathcal{O}_X) \to j_* K_2(\mathbf{C}(X)) \xrightarrow{T} \oplus_{x\in X}\, j_{x*}\mathbf{C}^*_x \to 0.$$

the equivalence of (1) and (3) follows by work of Bloch [B].

Now, there is the **Bloch-Beilinson conjectural filtration** on

$$CH^2(X,2) \otimes_\mathbf{Z} \mathbf{Q},$$

which has the form:

$$\begin{aligned} CH^2(X,2)\otimes_\mathbf{Z}\mathbf{Q} = \ & F^0CH^2(X,2)\otimes_\mathbf{Z}\mathbf{Q} \supseteq F^1CH^2(X,2)\otimes_\mathbf{Z}\mathbf{Q} \supseteq \\ & F^2CH^2(X,2)\otimes_\mathbf{Z}\mathbf{Q} \supseteq F^3CH^2(X,2)\otimes_\mathbf{Z}\mathbf{Q} = 0. \end{aligned}$$

Furthermore, we should have conjecturally:

CONJECTURE 2.1.

$$\begin{aligned} Gr^0CH^2(X,2)\otimes_\mathbf{Z}\mathbf{Q} &\cong \mathrm{Ext}^0_{\mathcal{MM}_\mathbf{C}}(\mathbf{Q}, H^2(X)(2)) \\ Gr^1CH^2(X,2)\otimes_\mathbf{Z}\mathbf{Q} &\cong \mathrm{Ext}^1_{\mathcal{MM}_\mathbf{C}}(\mathbf{Q}, H^1(X)(2)) \\ Gr^2CH^2(X,2)\otimes_\mathbf{Z}\mathbf{Q} &\cong \mathrm{Ext}^2_{\mathcal{MM}_\mathbf{C}}(\mathbf{Q}, H^0(X)(2)), \end{aligned}$$

where $\mathcal{MM}_\mathbf{C}$ is the conjectural category of mixed motives over $\mathbf{C}$.

We may note that since

$$H^2(X)(2) = \mathbf{Q}(2)$$

and

$$H^0(X)(2) = \mathbf{Q}(2),$$

and since

$$\begin{aligned} \mathrm{Ext}^0_{\mathcal{MM}_\mathbf{C}}(\mathbf{Q}, \mathbf{Q}(2)) &= 0, \\ \mathrm{Ext}^2_{\mathcal{MM}_\mathbf{C}}(\mathbf{Q}, \mathbf{Q}(2)) &\cong K_2(\mathbf{C}) \otimes_\mathbf{Z} \mathbf{Q}, \end{aligned}$$

we may simplify Conjecture 2.1 to the conjectural equalities:

CONJECTURE 2.2.

$$\begin{array}{rcl} Gr^0CH^2(X,2)\otimes_{\mathbf{Z}}\mathbf{Q} & \cong & 0, \\ Gr^1CH^2(X,2)\otimes_{\mathbf{Z}}\mathbf{Q} & \cong & \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{C}}}(\mathbf{Q},H^1(X)(2)), \\ Gr^2CH^2(X,2)\otimes_{\mathbf{Z}}\mathbf{Q} & \cong & K_2(\mathbf{C})\otimes_{\mathbf{Z}}\mathbf{Q}. \end{array}$$

Further, one expects that there is an injective map

$$\mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{C}}}(\mathbf{Q},H^1(X)(2))\to \mathrm{Ext}^1_{\mathcal{MHS}}(\mathbf{Z},H^1(X)(2))\otimes_{\mathbf{Z}}\mathbf{Q}\cong H^1(X,\mathbf{C}/\mathbf{Z}(2))\otimes_{\mathbf{Z}}\mathbf{Q},$$

where $\mathcal{MHS}$ is the category of mixed Hodge structures.

We expect that this map is induced by

$$R_X\otimes_{\mathbf{Z}}\mathbf{Q}\colon CH^2(X,2)\otimes_{\mathbf{Z}}\mathbf{Q}\to H^1(X,\mathbf{C}/\mathbf{Z}(2))\otimes_{\mathbf{Z}}\mathbf{Q};$$

indeed, we expect that:

CONJECTURE 2.3.

$$\mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{C}}}(\mathbf{Q},H^1(X)(2))\cong \mathrm{im}(R_X\otimes_{\mathbf{Z}}\mathbf{Q}).$$

It follows that

CONJECTURE 2.4. *Modulo torsion,*

$$\ker(R_X)=K_2(\mathbf{C}).$$

3. The Regulator of a very General Curve of Genus ≥ 2 is Torsion

The method employed here is the **infinitesimal invariant** of [G] and [V]. This allows us to use Hodge theory to reduce the problem to a new purely algebraic result, which is proved in section 4.

Let $p\colon \mathcal{U}\to\mathcal{M}$ be the **universal curve** over the moduli space of curves of genus g with a suitable level structure. If the regulator of a general curve has image which is non-torsion, one can find a family of elements $\beta(t)\in K_2(X_t)$, where t ranges over some generically finite branched cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$, with the property that

$$\nu(t)=R_{X_t}(\beta(t))$$

gives an analytic section of $R^1\tilde{p}_*(\mathbf{C}/\mathbf{Z}(2))\otimes_{\mathbf{C}}\mathcal{O}_{\tilde{\mathcal{M}}}$, where $\tilde{p}\colon\tilde{\mathcal{U}}\to\tilde{\mathcal{M}}$ is the lifting of the universal curve to $\tilde{\mathcal{M}}$, and such that $\nu(t)$ is non-torsion for a general t.

The first fact we need is the **infinitesimal property**:

PROPOSITION 3.1.

$$\nabla_{\tilde{\mathcal{U}}/\tilde{\mathcal{M}}}\nu(t)\in\Omega^1_{\tilde{\mathcal{M}}}\otimes F^1H^1(X,\mathbf{C}),$$

where $\nabla_{\tilde{\mathcal{U}}/\tilde{\mathcal{M}}}$ is the **Gauss-Manin connection**.

PROOF: This is well-known, but it is helpful to have a geometric proof.

If we have a local smooth family $p\colon\mathcal{X}\to B$ of curves over a 1-dimensional disc B, and $p^{-1}(t)=X_t$, then if we have a smoothly varying family of loops γ_t on X_t with base-point $p(t)$, and $F_\nu,G_\nu\in\mathbf{C}(\mathcal{X})^*$, let ρ be a path from t_0 to t_1 in B. We assume

$$\prod_\nu\{F_\nu,G_\nu\}\in\ker(T)=\bar{K}_2(X).$$

We want to compute

$$R_{X_{t_1}}(\prod_\nu\{F_\nu,G_\nu\})(\gamma_{t_1})-R_{X_{t_0}}(\prod_\nu\{F_\nu,G_\nu\})(\gamma_{t_0})$$

$$=\sum_\nu(\int_{\gamma_{t_1}}\log(F_\nu)\frac{dG_\nu}{G_\nu}-\log(G_\nu(p(t_1)))\frac{dF_\nu}{F_\nu}-\int_{\gamma_{t_0}}\log(F_\nu)\frac{dG_\nu}{G_\nu}-\log(G_\nu(p(t_0)))\frac{dF_\nu}{F_\nu}).$$

Let

$$U=\cup_{t\in\rho}\gamma_t$$

and

$$\sigma=\cup_{t\in\rho}p(t).$$

We cut U along σ and open it up to obtain $\tilde{U}$, whose boundary is $\gamma_{t_1},\gamma_{t_0},\sigma_1$, and σ_2,where the latter two paths are two copies of σ. $\tilde{U}$ is topologicaly a disc. and $\log(F)$, $\log(G)$ have well-defined branches on U. By Stokes' Theorem applied to $\tilde{U}$,

$$R_{X_{t_1}}(\prod_\nu\{F_\nu,G_\nu\})(\gamma_{t_1})-R_{X_{t_0}}(\prod_\nu\{F_\nu,G_\nu\})(\gamma_{t_0})$$

$$=\sum_\nu(\int_{\tilde{U}}\frac{dF_\nu}{F_\nu}\wedge\frac{dG_\nu}{G_\nu}+(\log(G_\nu(p(t_0)))-\log(G_\nu(p(t_1))))\int_{\gamma_{t_0}}\frac{dF_\nu}{F_\nu}+\int_{\gamma_{t_0}}\frac{dF_\nu}{F_\nu}\int_\sigma\frac{dG_\nu}{G_\nu}),$$

where we do not have problems in the interior of $\tilde{U}$ because any zeros and poles of the F_ν, G_ν cancel out because $\prod_\nu\{F_\nu,G_\nu\}\in\ker(T)$. The preceding formula reduces to

$$R_{X_{t_1}}(\prod_\nu\{F_\nu,G_\nu\})(\gamma_{t_1})-R_{X_{t_0}}(\prod_\nu\{F_\nu,G_\nu\})(\gamma_{t_0})=\sum_\nu\int_{\tilde{U}}\frac{dF_\nu}{F_\nu}\wedge\frac{dG_\nu}{G_\nu}.$$

Taking the limit as t_1 goes to t_0, we obtain

$$\nabla_{\mathcal{X}/B}R_{X_t}(\prod_\nu\{F_\nu,G_\nu\})(\gamma_t)=dt\otimes\sum_\nu\int_{\gamma_t}<\frac{dF_\nu}{F_\nu}\wedge\frac{dG_\nu}{G_\nu},\tau_t>$$

where $<,>$ represents contraction and τ_t is the normal vector of the variation of X_t in $\mathcal{X}$. Thus

$$\nabla_{\mathcal{X}/B}R_{X_t}(\prod_\nu\{F_\nu,G_\nu\})=dt\otimes\sum_\nu<\frac{dF_\nu}{F_\nu}\wedge\frac{dG_\nu}{G_\nu},\tau_t>\quad\in\Omega^1_B\otimes F^1H^1(X_t,\mathbf{C}).$$

This completes the proof of the Proposition.

Returning to the situation discussed at the beginning of this section, since

$$\nabla^2_{\tilde{\mathcal{U}}/\tilde{\mathcal{M}}}=0,$$

we have that

$$\nabla_{\tilde{\mathcal{U}}/\tilde{\mathcal{M}}}\nu(t)\in\ker(\Omega^1_{\tilde{\mathcal{M}}}\otimes F^1H^1(X_t,\mathbf{C})\xrightarrow{\nabla_{\tilde{\mathcal{U}}/\tilde{\mathcal{M}}}}\Omega^2_{\tilde{\mathcal{M}}}\otimes H^1(X_t,\mathbf{C})).$$

If we quotient out on the right from $H^1(X_t,\mathbf{C})$ to $H^{0,1}(X_t)$, then *a fortiori* we have

$$\nabla_{\tilde{\mathcal{U}}/\tilde{\mathcal{M}}}\nu(t)\in\ker(\Omega^1_{\tilde{\mathcal{M}}}\otimes F^1H^1(X_t,\mathbf{C})\xrightarrow{\nabla_{\tilde{\mathcal{U}}/\tilde{\mathcal{M}}}}\Omega^2_{\tilde{\mathcal{M}}}\otimes H^{0,1}(X_t)).$$

We make the identifications

$$\begin{array}{rcl} H^1(\Omega^1_{X_t}) & \cong & \mathbf{C} \\ \Omega^1_{\tilde{\mathcal{M}},t} & \cong & H^0(K_{X_t}^{\otimes 2}), \\ F^1H^1(X_t,\mathbf{C}) & \cong & H^0(K_{X_t}), \\ H^{0,1}(X_t) & \cong & H^0(K_{X_t})^*, \end{array}$$

and, if X_t is not hyperelliptic or $g = 1$, by Noether's Theorem

$$H^0(K_{X_t}^{\otimes 2}) = \frac{S^2 H^0(K_{X_t})}{W}.$$

where W is the quadratic ideal of the canonical image of X_t. Now

$$\nabla_{\tilde{\mathcal{U}}/\tilde{\mathcal{M}}}\nu(t) \in \ker(\frac{S^2V}{W} \otimes V \to \wedge^2(\frac{S^2V}{W}) \otimes V^*),$$

where $V = H^0(K_{X_t})$ and the map is induced by multiplication, as described in section 4. In section 4, we prove the general fact:

THEOREM 3.1. *Let V be a complex vector space of dimension $g \geq 2$, and*

$$W \subseteq S^2V$$

a linear subspace that does not contain any non-zero decomposable elements, i.e. images of elements of the form $v_1 \otimes_{\mathbf{C}} v_2$ with $v_1, v_2 \neq 0$. Then the natural map

$$\frac{S^2V}{W} \otimes V \to \wedge^2(\frac{S^2V}{W}) \otimes V^*$$

is injective.

In our case, since a decomposable element of W would be a union of two hyperplanes containing the canonical image of X_t, which cannot happen since the canonical curve is non-degenerate, the hypotheses of the theorem are satisfied and we conclude that

$$\nabla_{\tilde{\mathcal{U}}/\tilde{\mathcal{M}}}\nu(t) = 0.$$

We now employ a simple monodromy argument. Pick a general point $t_0 \in \tilde{\mathcal{M}}$. If

$$\rho\colon \pi_1(\tilde{\mathcal{M}}) \to \operatorname{Aut}(H^1(X_{t_0}, \mathbf{Z}))$$

is the **monodromy representation**, then for all loops γ in $\tilde{\mathcal{M}}$,

$$\rho(\gamma)(\nu(t_0)) - \nu(t_0) \in H^1(X, \mathbf{Z}(2)).$$

If we allow X_t to acquire a node, and γ is a loop around the value of t where this happens, then we have the **Picard-Lefschetz transformation** for any $v \in H^1(X, \mathbf{C})$,

$$\rho(\gamma)(v) = v + (v \cdot \delta)\delta,$$

where δ is the **vanishing cycle** associated to γ. It follows that

$$\nu(t_0) \cdot \delta \in \mathbf{Z}(2)$$

for all vanishing cycles δ. Since for the full family of curves of genus g, or a generically finite branched cover like $\tilde{\mathcal{M}}$, the vanishing cycles have finite index in $H^1(X, \mathbf{Z})$, it follows that $\nu(t_0) \in H^1(X, \mathbf{Q}(2))$. This says that

$$R_{X_{t_0}}(\beta(t_0)) \text{ is torsion.}$$

Since t_0 was a general point, this is a contradiction.

Appendix to §3: Proof of Theorem 1.1 This is similar to, but more elementary than, the proof of proposition 3.1 above.

Let B be a smooth algebraic curve and

$$B \to H^0(\mathcal{K}_2(\mathcal{O}_X))$$

an algebraic map. We want to show that the differential of the composite map

$$B \to H^0(\mathcal{K}_2(\mathcal{O}_X)) \xrightarrow{R_X} H^1(X, \mathbf{C}/\mathbf{Z})$$

is zero. If B is complete, this is clear since $H^1(X, \mathbf{C}/\mathbf{Z}(2)) \cong \oplus\mathbf{C}^*$, and any holomorphic map $B \to \mathbf{C}^*$ is constant.

In general, we let t be a local uniformizing parameter on B and give the above map as

$$t \to \alpha_t \to B_X(\alpha)$$

where

$$\alpha_t = \prod_\nu \{f_\nu(t),\ g_\nu(t)\}$$

with $f_\nu(t), g_\nu(t) \in \mathbf{C}(X)$ depending meromorphically on t. Then for $\gamma \in H_1(X, \mathbf{Z})$

$$\frac{d}{dt}R_X(\alpha_t)(\gamma) = \sum_\nu \int_\gamma (\frac{\dot{f}_\nu}{f_\nu}\frac{dg_\nu}{g_\nu} - \frac{\dot{g}_\nu}{g_\nu}\frac{df_\nu}{f_\nu})$$

and

$$R_X(\alpha_t)(\gamma) - R_X(\alpha_{t_0})(\gamma) = \sum_\nu \int_\gamma \int_{t_0}^t (\frac{\dot{f}_\nu}{f_\nu}\frac{dg_\nu}{g_\nu} - \frac{\dot{g}_\nu}{g_\nu}\frac{df_\nu}{f_\nu})dt.$$

We note that the integrand is a holomorphic 1-form on X and a logarithmic 1-form on the smooth compactification $\bar{B}$ of B. If μ is a closed loop in B

$$\sum_\nu \int_\gamma \int_\mu (\frac{\dot{f}_\nu}{f_\nu}\frac{dg_\nu}{g_\nu} - \frac{\dot{g}_\nu}{g_\nu}\frac{df_\nu}{f_\nu})dt \in \mathbf{Z}(2),$$

i.e., by integration the above map is

$$B \to H^0(\Omega^1_{X/\mathbf{C}})/H^1(X, \mathbf{Z}(2)) \subseteq H^1(X, \mathbf{C}/\mathbf{Z}(2)).$$

Now, as noted above

$$\sum_\nu (\frac{\dot{f}_\nu}{f_\nu}\frac{dg_\nu}{g_\nu} - \frac{\dot{g}_\nu}{g_\nu}\frac{df_\nu}{f_\nu})dt \in H^0(\Omega^1_{X/\mathbf{C}}) \otimes \Omega^1_{\bar{B}/\mathbf{C}}(\log D)$$

for some set of points D on $\bar{B}$, and for $t \in \bar{B}\backslash D$

$$\sum_\nu \int_\mu (\frac{\dot{f}_\nu}{f_\nu}\frac{dg_\nu}{g_\nu} - \frac{\dot{g}_\nu}{g_\nu}\frac{df_\nu}{f_\nu})dt \in H^0(\Omega^1_{X/\mathbf{C}}) \cap H^1(X, \mathbf{Z}(2)) = 0.$$

Thus

$$B \to H^0(\Omega^1_{X/\mathbf{C}})$$

$$t \mapsto \sum_\nu \int_{t_0}^t (\frac{\dot{f}_\nu}{f_\nu}\frac{dg_\nu}{g_\nu} - \frac{\dot{g}_\nu}{g_\nu}\frac{df_\nu}{f_\nu})dt$$

is well-defined, and since the integrand has at most logarithmic singularities and no residues it extends to $\bar{B}$, and therefore must be constant.

4. An Algebraic Result

Given a vector space V and a linear subspace $W \subseteq S^2V$, the multiplication map

$$V \otimes V \to S^2V$$

induces a map

$$V \to S^2V \otimes V^*$$

and hence a map

$$V \to \frac{S^2V}{W} \otimes V^*.$$

By tensoring with $\mathrm{id}_{S^2V/W}$, we obtain a natural map

$$\frac{S^2V}{W} \otimes V \to \frac{S^2V}{W} \otimes \frac{S^2V}{W} \otimes V^*$$

which then gives a natural map

$$\frac{S^2V}{W} \otimes V \to \wedge^2(\frac{S^2V}{W}) \otimes V^*.$$

THEOREM 4.1. *Let V be a complex vector space of dimension $g \geq 2$, and*

$$W \subseteq S^2V$$

a linear subspace that does not contain any non-zero decomposable elements, i.e. images of elements of the form $v_1 \otimes_{\mathbf{C}} v_2$ with $v_1, v_2 \neq 0$. Then the natural map

$$\frac{S^2V}{W} \otimes V \to \wedge^2(\frac{S^2V}{W}) \otimes V^*$$

is injective.

PROOF: Assume that

$$\alpha \in \frac{S^2V}{W} \otimes V$$

maps to 0 under the map above. We may think of α as a map

$$V^* \to \frac{S^2V}{W},$$

and let r be the rank of this map. Choose a basis

$$e_1^*, \dots, e_g^*$$

for V^* such that

$$e_{r+1}^*, \dots, e_g^*$$

span $\ker(\alpha)$. Let

$$e_1, \dots, e_g$$

be the dual basis for V. We may write

$$\alpha = \sum_{i=1}^{r} \phi_i \otimes e_i \in \frac{S^2V}{W} \otimes V.$$

By construction, $\phi_1, \dots, \phi_r$ are linearly independent; let

$$U \subseteq \frac{S^2V}{W}$$

denote the subspace they span. Under the map of the theorem, α maps to

$$\sum_{i=1}^{r}\sum_{j=1}^{g}(\phi_i \wedge e_i e_j) \otimes e_j^* \in \wedge^2(\frac{S^2V}{W}) \otimes V^*.$$

Thus, for all j,

$$\sum_{i=1}^{r} \phi_i \wedge (e_i e_j) = 0.$$

If

$$e_i e_j \notin U,$$

then no term of the sum can cancel $\phi_i \wedge (e_i e_j)$. If

$$P = \mathrm{span}(e_1, \dots, e_r) \subseteq V,$$

then we conclude that

$$P \otimes V \to U$$

under the multiplication map

$$V \otimes V \to \frac{S^2V}{W}.$$

By hypothesis on W, no tensor product of non-zero elements $v_1 \otimes v_2$ maps to 0. By the Lemma below, we conclude that

$$\dim(P) + \dim(V) - 1 \leq \dim(U),$$

i.e.

$$r + g - 1 \leq r$$

and hence $g \leq 1$. This contradicts the hypothesis $g \geq 2$, and hence α must be 0.

LEMMA 4.1. *Let A, B, C be finite-dimensional vector spaces over $\mathbf{C}$, and*

$$F \colon A \otimes_{\mathbf{C}} B \to C$$

a linear map such that

$$F(a \otimes b) \neq 0$$

for all non-zero elements a, b. Then

$$\dim(A) + \dim(B) - 1 \leq \dim(C).$$

This is classical and appears for example in the proof of Clifford's Theorem.

5. The Image of R_X Includes all Torsion Points

Let X be a smooth projective curve of genus g. Our proof proceeds using classical complex methods from Riemann surface theory, where we save some terms that are ordinarily thrown away. The first step is to open up X into a fundamental domain U whose sides are $a_1, b_1, \dots, a_g, b_g$. Pick p_0 the first vertex of the domain. Let $\omega_1, \dots, \omega_g$ be a basis for $H^0(\Omega^1_{X/\mathbf{C}})$ and

$$F_i(p) = \int_{p_0}^{p} \omega_i.$$

Let $F(p)$ be the corresponding vector $(F_1(p), \ldots, F_g(p)) \in \mathbf{C}^g$. By the implicit function theorem, for any vector $\xi \in \mathbf{C}^g$, there exist for k large enough points $p_1, \ldots, p_k$ in the interior of U and integers $j_1, \ldots, j_k$ such that

$$\sum_{\nu=1}^{k} j_\nu F(p_\nu) = \xi.$$

Now choose an arbitrary element of the integral lattice

$$\sum_{i=1}^{g} m_i a_i^* + n_i b_i^*$$

in $H^1(X, \mathbf{Z})$, where $a_1^*, b_1^*, \ldots, a_g^*, b_g^*$ is the dual basis to $a_1, b_1, \ldots, a_g, b_g$, and an arbitrary positive integer N. We may choose a collection of points $p_1, \ldots, p_k$ in the interior of U and integers $j_1, \ldots, j_k$ such that

$$\sum_{\nu=1}^{k} j_\nu F(p_\nu) = \sum_{i=1}^{g} \frac{m_i}{N} a_i^* + \frac{n_i}{N} b_i^*.$$

If

$$D = \sum_{\nu=1}^{k} j_\nu p_\nu$$

as a divisor, then by the Abel-Jacobi theorem, there exists $f \in \mathbf{C}(X)^*$ such that

$$\operatorname{div}(f) = ND.$$

If we do a contour integral on U for the 1-form $F_j df/f$, we obtain the

$$\sum_{i=1}^{g} \int_{a_i} \omega_j \int_{b_i} \frac{df}{f} - \int_{b_i} \omega_j \int_{a_i} \frac{df}{f}) = 2\pi i F_j(ND).$$

It follows that

$$\sum_{i=1}^{g} \phi(a_i^*) \int_{b_i} \frac{df}{f} - \phi(b_i^*) \int_{a_i} \frac{df}{f} = 2\pi i F(ND) = 2\pi i \sum_{i=1}^{g} m_i \phi(a_i^*) + n_i \phi(b_i^*),$$

where

$$\phi\colon H^1(X, \mathbf{Z}) \to \frac{H^1(X, \mathbf{C})}{F^1 H^1(X, \mathbf{C})} \cong \mathbf{C}^g$$

is induced by the coefficient map. Since ϕ maps $H^1(X, \mathbf{Z})$ injectively to $\mathbf{C}^g$, it follows that

$$\int_{a_i} \frac{df}{f} = -2\pi i n_i,$$

$$\int_{b_i} \frac{df}{f} = 2\pi i m_i.$$

Now let

$$\mu = e^{\frac{2\pi i}{N}}.$$

Consider the element

$$\{\mu, f\} \in K_2(\mathbf{C}(X)).$$

We have that

$$T(\{\mu, f\}) = \sum_{\nu=1}^{k} (\mu^{N j_\nu})_{p_\nu} = \sum_{\nu=1}^{k} (1)_{p_\nu} = 0.$$

Thus

$$\{\mu, f\} \in K_2(X).$$

The regulator is computed by

$$R_X(\{\mu, f\})(a_i) = \int_{a_i} \log(\mu) \frac{df}{f} = \frac{2\pi i}{N}(-2\pi i n_i) = -(2\pi i)^2 \frac{n_i}{N} \in \mathbf{Q}(2)$$

and similarly

$$R_X(\{\mu, f\})(b_i) = (2\pi i)^2 \frac{m_i}{N} \in \mathbf{Q}(2).$$

Since n_i, m_i were arbitrary integers and $N > 0$ was arbitrary, we conclude that every element of $H^1(X, \mathbf{Q}(2)/\mathbf{Z}(2))$ can be realized by an element of $K_2(X)$ of this form. This proves that the image of R_X contains all of the torsion points.

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