REMARKS ON IGUSA THEORY AND REAL ORBITAL INTEGRALS

ROBERT P. LANGLANDS

Contents

The study of orbital integrals on p-adic groups has turned out to be singularly difficult, and even the most basic results in the simplest examples are surprisingly hard to come by ([H]). One technique is derived from methods introduced by Igusa and expounded by him in the notes [I]. For real groups there is a much more incisive technique that is based on the differential equations exploited by Harish-Chandra to study the behaviour of orbital integrals near singular elements ([HC I]), and that Shelstad used to obtain the first serious results on transfer and endoscopy. (See [S], as well as other papers by the same author.) They are much more general than those that can be obtained with the Igusa theory so that there is little incentive to develop it for real groups, where it is more complicated, the description of the asymptotic behaviour of the functions appearing being of necessity much more elaborate. None the less for specific purposes in specific contexts it can be more efficient than Shelstad's general methods. For example, the detailed analysis of the stabilization of the trace formula carried out in [R] requires an exact calculation of the constants that express the unipotent orbital integrals of the transferred function in terms of those of the original function. For p-adic groups the calculation is carried out in [SL II], but for real groups it is not available. Since [R] is a prerequisite to this volume, it was felt that it would be useful to include the necessary calculation, and it appears that it is made most simply with the help of Igusa theory, for once the basic facts are reviewed we can imitate the second appendix of [SL II].

Since this note, itself, is no more than an appendix to [SL I] and [SL II], we use the notation and results of these two papers freely. In particular, we assume that the reader is familiar with the notion of transfer factor and with transfer of functions by orbital integrals.

Igusa theory

The first step is to state and prove the analogue of Proposition 1.1 of [SL I]. If F is a function of the real or complex variable x in a neighbourhood of $x = 0$ then

Appeared in *The zeta functions of Picard modular surfaces*, ed. Robert P. Langlands and Dinakar Ramakrishnan, Les Publications CRM Montreal (1992).

the statement

$$F(x) \sim \sum_\beta \sum_\theta \sum_r F(r,\theta,\beta) \log\bigl(|x|\bigr)^{r-1} \theta(x)|x|^{\beta-1}, \tag{1.1}$$

where $F(r,\theta,\beta)$ is a constant and where for a given β the sums over θ and r are finite, will mean that if a real number B is given then the difference between $F(x)$ and the sum of all the terms on the right for which $\beta < B$ is of the order of $|x|^{B'}$ for any B' less than B. The symbol θ denotes a character of absolute value 1 that is identically 1 on the positive real numbers and r a positive integer.

The hypotheses are the same as those of the original proposition except that the base field is now archimedean and

$$f(y) = \gamma(y)\kappa_1(\mu_1)\cdots\kappa_n(\mu_n)$$

where γ is no longer a constant but a smooth function of y.

Take F to be of compact support and consider its Mellin transform.

$$\Theta(s) = \Theta(s,\theta) = \int F(x)\theta^{-1}(x)|x|^s\,dx.$$

There are two provisional hypotheses. Suppose first of all that for a given B and for some finite set T of θ,

$$\sum_{\theta\notin T} \int_{|z|=1} F(xz)\theta^{-1}(z)\left|\frac{dz}{z}\right|,$$

is $o\bigl(|x|^B\bigr)$. Suppose, furthermore, that for any θ the function $\Theta(s,\theta)$ is $o\bigl(|s|^{-N}\bigr)$ for all real N and on any vertical strip of finite width and that in any such strip it has only a finite number of poles. Granted these hypotheses, we easily establish the asymptotic expansion (1.1).

If

$$F(x,\theta) = \int_{|z|=1} F(xz)\theta^{-1}(z)\left|\frac{dz}{z}\right|,$$

then

$$|x|\theta(x)F(x,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Theta(s,\theta)|x|^{-s}\,ds, \tag{1.2}$$

provided that $c \gg 0$. This is an immediate consequence of the formula of Theorem 9.a of [W]. Indeed, if $u = |x|$ then

$$dx = du\,|dz/z|, \qquad \text{(real field)},$$
$$dx = |dz\wedge d\overline{z}| = du\,|dz/z|, \qquad \text{(complex field)}.$$

Consequently,

$$\Theta(s,\theta) = \int_0^\infty F(u,\theta)u^s du.$$

Applying the theorem cited, one obtains (1.2).

Observe also that

$$F(x) = \left(\int_{|z|=1}\left|\frac{dz}{z}\right|\right)^{-1} \sum F(x,\theta),$$

so that any statement about the asymptotic behaviour of the $F(x,\theta)$ translates immediately, in view of the provisional hypotheses, into a statement about the asymptotic behaviour of $F(x)$.

The vertical line over which the integral of (1.2) is taken can be moved further and further to the left until it finally yields a term that is $o(|x|^B)$. This leaves the residues that are clearly of the form

$$\sum_r D^r\left(|x|^{-s}\right), \qquad (D = d/ds),$$

so that we have the required asymptotic expansion.

Returning to the two provisional hypotheses and the paper [SL I], I observe first there is an obvious misprint in the definition of $\Theta(s)$ on page 474 and note also that a γ has gone astray on line 3 of page 475. This admitted, the formula for $\Theta(s,\theta)$, or rather for the contribution to it of one element of a partition of unity, that is given at the top of page 475 becomes in the archimedean case the integral over the set

$$|\mu_i| < \epsilon_i, \qquad 1 \leqslant i \leqslant n, \tag{1.3}$$

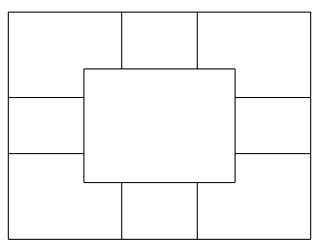
of the product of two terms:

$$\left|\alpha(y)\right|^s \theta\left(\alpha(y)\right)^{-1}\gamma(y); \tag{1.4}$$

and

$$\prod_i \Bigl(\kappa_i(\mu_i)\theta^{-1}(\mu_i)|\mu_i|^{a_i s+b_i-1}\Bigr). \tag{1.5}$$

The function γ may be taken to have compact support in the region of integration and $\alpha(y)$ to be constant (see [I, p. 99]) so that (1.4) is smooth and of compact support. We then decompose the box (1.3) as in the diagram, so that in each of the sub-boxes γ is approximated to a very high order by its Taylor expansion in the coordinates that do not vanish in the box. (In the complex case this Taylor expansion involves μ_i and $\overline{\mu}_i$.)



Thus inside one of these boxes it has an expansion to arbitrarily high order as

$$\sum a \prod \mu_i^k \qquad \text{(real field)},$$
$$\sum a \prod \mu_i^k \overline{\mu}_i^\ell \qquad \text{(complex field)},$$

the coefficients being smooth functions of the μ_i and $\overline{\mu}_i$ that do not vanish in the box under consideration. It is proven in [I, §III.4] that the integrals over each of the boxes have the required behaviour as a function of s, so that there is no problem with Proposition 1.1. The point, however, is so to define principal values that Proposition 1.2 is valid.

What has to be defined is

$$\mathrm{PV}\int_{D_r\cap U} h_r|\nu_r|,$$

where the symbols D_r, h_r, and ν_r have the same meaning as on line -7 of page 469.

The first step is, however, to define the set $E(\theta,\beta)$ that occurs on line 5 of page 468. If the field is real let η be the sign character and if the field is complex let

$$\eta(z)=\frac{z}{|z|},\quad \overline{\eta}(z)=\eta(\overline{z}).$$

Then $E(\theta,\beta)$ is the set of all divisors and all non-negative integers (real field) or pairs of non-negative integers (complex field) such that

$$\begin{aligned} k+b(E)&=\beta a(E),\quad \theta^a\kappa=\eta^k &&\text{(real field)},\\ k+\ell+b(E)&=\beta a(E),\quad \theta^a\kappa=\eta^k\overline{\eta}^\ell &&\text{(complex field)}.\end{aligned}\tag{1.6}$$

Notice that for a given $b(E)$ and $\kappa=\kappa(E)$, as well as a given θ and β, these conditions determine k or k and ℓ.

We next consider the conditions (i) to (iv) on page 469 observing first that s has been given two meanings in the paper. (For clarity, I now denote the integer introduced in condition (i) by an upper-case letter.) The integer m_i is replaced by δ_i, the side of the box, and M by $\delta=|\alpha|$. The third and fourth conditions are clearly too strong for the archimedean case; all we demand is that both w and γ have Taylor expansions to arbitrarily high order in the variables μ_i that vanish in the box.

To proceed further we introduce the analogue of

$$(1-1/q)^{(S-1)}A_r(m)$$

on page 470, but we first observe, with the proof in mind, that it is the asymptotic behaviour of $F(x,\theta)$ that we are trying to determine. Since

$$\int_0^\epsilon |x|^\beta \log|x|^r|x|^{s-1}\,dx,$$

is equal to

$$D^r(\epsilon^{s+\beta}/s+\beta),\quad D=d/ds,$$

its principal part at $s=-\beta$ is that of

$$D^r\big(1/(s+\beta)\big)=(-1)^r r!\,(s+\beta)^{-(r+1)}.$$

Thus in calculating $F(r,\theta,\beta)$ we are trying to calculate $(-1)^r(1/r!)$ times the coefficient of $(s+\beta)^{-(r+1)}$ in the principal part of

$$\left(\int\left|\frac{dz}{z}\right|\right)^{-1}\Theta(s,\theta)\tag{1.7}$$

at $-\beta$.

As on page 475 of [SL I] we shall use the continuity of the principal part to reduce first to the case that the product of w and γ is a monomial $m(y)$ given by

$$\begin{aligned} &\mu_1^{k_1}\cdots\mu_t^{k_t}, &&\text{(real field)},\\ &\mu_1^{k_1}\overline{\mu}_1^{\ell_1}\cdots\mu_t^{k_t}\overline{\mu}_t^{\ell_t}, &&\text{(complex field)}.\end{aligned}$$

here t is the number of coordinates that vanish in the box. For the same reasons, again as in [SL I], we shall also be able to disregard the integration with respect to the coordinates that do not vanish. It is, for example, incorporated in a trivial way in the principal value on the right of the equation on line 8 of page 470.

To calculate (1.7) we take the integral over (1.3) of the product of (1.4) and (1.5). The integration with respect to the μ_i, $i>t$, we ignore for the reasons given in the last paragraph. The integration with respect to the other variables is 0 unless the second of the conditions (1.7) is satisfied for all the divisors $\mu_i = 0$, $1 \leqslant i \leqslant p$, but if that condition holds then we obtain a product

$$|\alpha|^s \left(\frac{\delta_i^{(q_i+a_is+b_i)}}{q_i + a_is + b_i} \right) \int \left| \frac{dz}{z} \right|,$$

where q_i equals k_i if the field is real and $k_i + \ell_i$ if it is complex. The number δ_i is the side (or perhaps better the radius) of the pertinent box in the partition.

It follows immediately from this that the factor

$$(1 - 1/q)^{S-1}$$

that appears on line 8 of page 470 is to be replaced by

$$\left(\int \left| \frac{dz}{z} \right| \right)^{S-1}.$$

Moreover $A_r(m)$ is the coefficient of z^{-r} in the Laurent expansion of

$$|\alpha|^z \prod_{i=1}^{s} (\delta_i^z / a_i z), \tag{1.8}$$

divided by $(-1)^r r!$. The argument here is a little simpler than that of [SL I] because in that paper I expand (1.8) twice, first on page 470 and then on page 476.

The term $h_s|\nu_s|$ appearing in the integration on line 8 of page 470 has to be changed. The factor

$$w(0, \dots, 0, \mu_{s'}, \dots, \mu_n)\gamma(0, \dots, 0, \mu_{s'}, \dots, \mu_n),$$

where $s' = S + 1$, that occurs in the product of h_S and μ_S (see the bottom of page 468) has to be modified by the action of

$$\prod_{i=1}^{S} \frac{D_i^{ki}}{k_i!} \qquad \text{(real field)},$$

$$\prod_{i=1}^{S} \frac{D_i^{k_i}}{k_i!} \frac{\overline{D}_i^{\ell_i}}{\ell_i!} \qquad \text{(complex field)}.$$

It is best of course to verify that the principal values introduced here are well-defined, but unnecessary for the limited purposes to which we will put the theory. It is also inconvenient to suppose that α is a constant, but provided that, as here, we are only dealing with $k = 0$ and $\ell = 0$ this is no problem.

Applications

In the archimedean case as in the nonarchimedean, results of Harish-Chandra assure us that the logarithmic terms do not appear when treating orbital integrals. For the archimedean case consult [HC I].

For the unitary group in three variables, SU(3), only the real field is of any interest because transfer of orbital integrals is easy over the complex field. Hence we take the base field to be **R**. We also take the group to be quasisplit. As Rogawski observes [R, Prop. 4.9.1] the existence of the transfer f^H to U(2) of a function f on SU(3) is an immediate consequence of results of Shelstad and Clozel-Delorme. Since the group is quasisplit, we may choose the transfer factor Δ_0 as in §3.7 of

[TF I]. Since the order of every element in the group $H^1(Z_{\rm sc})$ appearing in §5.1 of that paper is divisible by 3, while the element $\mathbf{s}_T$ defining the group H is of order 2, the value of the pairing $\big\langle \mathrm{inv}_T(u), \mathbf{s}_T \big\rangle$ is 1 and so is the invariant $\Delta(u)$. Thus, by [TF I, Theorem 5.5.A], we have an equality of integrals over the regular unipotent classes in the two groups with respect to the measures of [TF I], §5.1.

$$\int f = \int f^H.$$

Our concern here is, however, with the subregular orbital integral,

$$\int f\Big(n_\infty^{-1} n(w) n_\infty\Big)|w|\,dw\,dn_\infty \tag{2.1}$$

of [SL II, Appendix II]. I claim that if the transfer factor is taken to be Δ_0, then it is equal to $f^H(1)$.

We examine, in two different ways, the asymptotic behaviour of

$$D_H(\gamma_H)\Phi^{\rm st}(\gamma_H, f^H), \tag{2.2}$$

treating it first as defined by orbital integrals on $H = \mathrm{U}(2)$. The pertinent reference is [PV]. The archimedean Igusa theory can be applied just as the nonarchimedean theory was. For our present purposes the relevant terms from (1.1) are those for which $\beta = 2$.

Consider the equations (1.6) and (3.12), (3.14) of [PV]. Then

$$k + b(E) = 2 \implies k = 0,\ E = E_1 \quad \text{or} \quad k = 1,\ E = E_2. \tag{2.3}$$

The second is pertinent only if

$$\theta\kappa = \eta. \tag{2.4}$$

This of course is a possibility, but not one that concerns us here. We are dealing with the stable orbital integral, so that, by (3.17) of [PV], $\kappa = 1$, and $\theta = \eta$. Thus if we confine attention to $\theta = 1$, as we shall, then we may take $k = 0$.

Since k is now 0, the integer $S = 1$, and $a(E) = 1$, we conclude that the formula (4.5), with $\kappa = 1$, remains the value for Λ_1 in (4.1) of [PV], although the right side of formula (4.1) is no longer correct as it stands, for it now contains just two terms of the asymptotic expansion for $\Gamma_1^{\rm st}$ of [SL II, Lemma 4.2] and for the complete expansion must be supplemented by others, that are, however, of higher order and not pertinent here.

Since the orbital integrals of f and of f^H are related as in §1.4 of [TF II] by the formula

$$\Phi^{\rm st}(\gamma_H, f^H) = \sum_\gamma \Delta(\gamma_H, \gamma)\Phi(\gamma, f)$$

the asymptotic behaviour of (2.4) can be examined on SU(3) as in [SL II, Appendix II]. The first step is once again to decide which values of k and which divisors yield $\theta = 1$, $\beta = 2$. The equation

$$k + b(E) = 2a(E),$$

together with (4.7) and (4.8) of [SL I] provide the following information.

- E_1', E_2'': $k = 0$.
- E_2^1, E_2^2, E_2^3: excluded.
- E_3: excluded.

- E_4: $k = 1$, however this is the regular unipotent element and in [SL II, §5] it is shown although never mentioned explicitly that for the divisor corresponding to this orbit κ is always 1. Thus the relation (2.4) excludes this case.
- E_5: excluded.
- E_6: $k = 0$, because $\frac{b(E)}{a(E)} = 2$.

As in the first paragraph of [SL II, §6] the orbits E_1', E_2'' have no rational points and may be ignored. Thus we have only E_6 to deal with, and $S = 1$.

Since $k = 0$, no derivatives are involved and all formulas from [SL II] remain valid for the real field, except perhaps that for the constant $A_r(M)$, whose definition becomes slightly different. The integer r is equal to 1. For a p-adic field, the value of $A_r(M)$ was calculated at the end of §9 of [SL II] and found to be 1/2. For the real field the same result is valid because, in the present circumstances, the expression (1.8) becomes

$$\frac{\big(\delta_i|\alpha|\big)^z}{2z},$$

which has the same principal part at 0 as $1/2z$.

The calculation of §9 of [SL II] again shows that the constant a of Lemma 4.2 of that paper is $\big[E^1 : (E^1)^2\big]$, which is clearly equal to 1 when $E = \mathbf{C}$. We conclude that

$$\int f\Big(n_\infty^{-1} n(w) n_\infty\Big)|w|\,dw\,dn_\infty = f^H(1).$$

There is a final observation that is needed in [R], and for all local fields of characteristic 0. As in [SL II] take the local field to be F and the quadratic extension used to define the unitary group to be E. Take η to be the character of $F^\times$ defined by E. Choose any element w_0 of E such that $w_0 + \overline{w}_0 = 0$. Then the integral

$$\int f\Big(n_\infty^{-1} n(w) n_\infty\Big)\eta\Big(\frac{w}{w_0}\Big)|w|\,dw\,dn_\infty \tag{2.5}$$

is defined.

A stable distribution on $G(F)$ is a distribution that can be approximated by the distributions

$$f \to \Phi^{\mathrm{st}}(\gamma, f),$$

appearing in Section 1.4 of [TF I]. The final observation is that the distribution (2.5) is stable. It is clear that the coefficients appearing in the germ-expansions of stable orbital integrals, or, if the field is archimedean, in the more complicated expansions of this note, are stable. We shall consider, therefore, the expansion of $D_G(\gamma)\Phi^{\mathrm{st}}(\gamma, f)$ near the identity, examining the term corresponding to $\beta = 2$. Rather than taking $\theta = 1$ we shall take $\theta = \eta$, but some preliminary observations are necessary.

In general, we could consider

$$D_G(\gamma)\Phi(\gamma, f). \tag{2.6}$$

Since they are finite linear combinations of κ-orbital integrals, the Igusa theory can be applied to them. In particular, the terms in their expansions corresponding to $\beta = 2$ and $\theta = \eta$ are linear combinations of the corresponding terms in the κ-orbital integrals. For p-adic fields, therefore, it is a consequence of [HC II, Lemma 24], which was drawn to my attention by J. Rogawski, that both of the integrals (2.1) and (2.5) must be linear combinations of the terms with $\beta = 2$ in the expansion of κ-orbital integrals. For real groups, this is a consequence of [B, Th. 6.7].

The results of §8 of [SL II] show that, the κ-orbital integrals attached to endoscopic groups that are tori contribute nothing to the terms with $\beta = 2$, and those results together with those above show that the κ-orbital integrals attached to the endoscopic group H yield the integral (2.1). Since there are two conjugacy classes of subregular elements in SU(3), the terms with $\beta = 2$ in the stable orbital integral must be different from zero for at least one torus, which will clearly have to be anisotropic. Consequently it suffices to show that for any anisotropic torus these terms are a multiple of (2.5); there is no need to show that the coefficient is not 0. That might, indeed, be a disagreeable task.

The expansion of the stable orbital integral over a p-adic field can be performed as in [SL II], and we consider only these fields, but as we have already seen, the calculations over the real field are essentially the same. Since this note is short and we are not pressed for space, we first treat tori that are contained in proper endoscopic groups. This permits a less abrupt effort at recollection of the notation of [SL II]. Suppose first that the endoscopic group is H. The pertinent formulas are on p. 500, but since we now take κ to be 1, the cocycle given by (8.11) is irrelevant, and the integrand is (8.10).

Since the integrand reduces to (8.10), it is identically 1 if $\theta = 1$ and the analogue of (8.12) becomes

$$\oint |dV_1| = 0.$$

Thus θ is η. Since U and V are linear functions of V_1, a straightforward calculation shows that the argument of θ^{-1} in (8.10) is equal to

$$\frac{a^2c}{d^2b}(cV_1 - bw)(bV_1 + bw)V_1.$$

Replacing w by tw with $t \in F$, we multiply it by t^3, because V_1 is also multiplied by t. Thus the integrand is multiplied by $\theta(t)$. It follows readily that the term in the expansion corresponding to $\beta = 2$, and $\theta = \eta$ is a multiple of (2.5).

When the endoscopic group is an anisotropic torus, the argument is similar. We again apply formulas from [SL II], although as observed in the remarks concluding this note, this should be done only with caution. We begin by remarking that if the endoscopic group is an anisotropic torus and if ρ and σ and y_0 are chosen as in the discussion before and after (8.5), then once again the term corresponding to $\beta = 2$ and θ is 0 unless $\theta = \eta$.

Take, therefore, $\theta = \eta$. Then, again with the notation following (8.5), we write $y = ty_0Rr$, so that the argument of θ^{-1} becomes

$$y_0^2R^2\frac{V}{U}(V-1)w. \tag{2.7}$$

Recall that (6.5) provides an identification of the base I_n with the $\mathbf{P}_1$ defined by the elements of trace 0 in the fixed field of ρ. With this in mind, it is convenient to write (2.7) as cw times

$$\frac{(y_0R)(y_0RV)\big(y_0R(1-V)\big)}{dy_0R(V-1) - by_0RV}. \tag{2.8}$$

The three factors in the numerator are simply, apart perhaps from sign, the three coordinates of an element of trace 0 in the cubic field over E attached to the torus. Multiplying this element by a scalar from F we multiply (2.8) by the scalar squared and do not change the value on it of θ, extended to E or to an even larger field if necessary. The conclusion is that the value of θ on (2.8) is a perfectly good integrand on the base I_n that has nothing to do with w and that the dependence on w is all through $\theta^{-1}(cw)$. The conclusion is, once again, the term in the expansion corresponding to $\beta = 2$, and $\theta = \eta$ is a multiple of (2.5).

In general the map

$$V \to (-1, V, 1 - V)$$

sends the base I_n to a twisted form of the $\mathbf{P}_1$ over E attached to vectors of trace 0 in the cubic extension K of E defined by the torus. In concrete terms, the unitary group is defined by an equation,

$$J = \overline{A}^t JA,$$

and the field K is the commutant of the torus in $\mathrm{GL}(3, E)$. It is provided with an involution,

$$A \to J\overline{A}^t J^{-1}$$

The image of I_n is the $\mathbf{P}_1$ over F defined by the elements of trace 0 fixed by the involution.

If we observe that the quotient of the values attached to any two opposite rays in Figure 6.1 is $-z/(V-1) = -z_1$ or its inverse, we conclude readily that $z_1 \in E$ and that $\overline{z}_1 = z_1^{-1}$. Choose r such that $\overline{r} = z_1 r$. Then the pertinent part of Figure 6.1 is reproduced in Figure A.

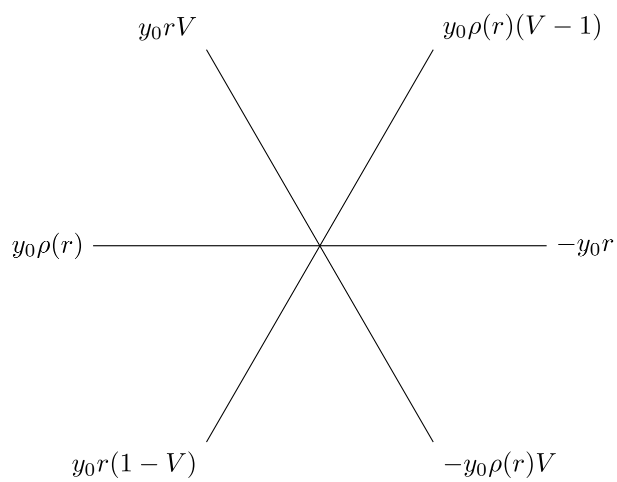


Figure A

In this diagram x has been taken equal to $yz_1(V-1)$ and y to $y_0 r$. The rationality condition,

$$\epsilon(\sigma_T)\sigma(x_{\sigma_T^{-1}s}) = x_s,$$

that appears on p. 490 is simply the condition that $\left\{-ay_0, ay_0V, ay_0(1-V)\right\}$ is an element in K fixed by the involution, or rather its three eigenvalues with respect to some diagonalization of T and K. If δ is the homomorphism of the extended Weyl group into $\{\pm 1\}$ that is equal to ϵ on the ordinary Weyl group and to $-\epsilon$ on its complement then a is so chosen that $\sigma(a) = \delta(\sigma_T)a$ for all σ.

Thus it is clear that once one y_0 is chosen all the rest are equal to ty_0 with $t \in F$. Once again, we apply the formula for λ on p. 497, and come to the same conclusion about θ, first that it is equal to 1 or to η and then that it is equal to η. We then have as argument of θ^{-1} the product of w with

$$\frac{y_0^2 V(V-1)}{U}$$

and this is c/a^2 times

$$\frac{(ay_0)(ay_0V)\big(ay_0(V-1)\big)}{day_0(V-1)-bay_0V}.$$

The argument can now precede as before.

I add a few remarks about the formulas in [SL II], observing in passing that the same symbols are often used there for quite different objects. The symbols x, y, and z play in particular several different roles that are, however, easily distinguished. Moreover the R and the V_1 on p. 498 are not the same as those on p. 500, and are indeed not only largely superfluous but in addition badly chosen because, so far as I can now see, $\rho(V_1) = -V_1$. Fortunately, this requires only inessential modifications, that I forgo. There are also several obvious misprints in the middle of the page, and a bar missing in the condition for rationality of U_1 on p. 493.

The difficulties of manipulating easily the rationality conditions that arise in the Igusa theory and of expressing them clearly suggest either that the theory is a clumsy tool that should be abandoned or that, like endoscopy itself, it is much deeper than appears at first glance, and deserves a certain amount of study for its own sake. At the moment I incline to the second view. The work of T. Hales provides persuasive evidence for it. The above discussion also provides some hope that the rationality conditions may not be so opaque as they seem.

References

[B] D. Barbasch, *Fourier inversion for unipotent invariant integrals*, Trans. of the AMS, vol. 249, (1979) 51–83.

[H] T. Hales, *Orbital integrals on* U(3), this volume, pp. 303–334. [The zeta functions of Picard modular surfaces, ed. Robert P. Langlands and Dinakar Ramakrishnan, Les Publications CRM Montreal (1992).]

[HC I] Harish-Chandra, *A formula for semisimple Lie groups*, Amer. J. Math. vol. 79 (1957) 733–760.

[HC II] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, Proceedings of the 1977 Annual Seminar of the Canadian mathematical Congress, ed. W. Rossmann, Queen's Papers in Pure and Applied Math. **48**, (1978) 281–347.

[I] J. I. Igusa, *Lectures on forms of higher degree*, Tata Institute of Fundamental Research, Bombay (1978).

[SL I] R. P. Langlands, *Orbital integrals on forms of* SL(3) I, Amer. Jour. Math., **105**, (1983) 465–506. [doi:10.2307/2374265.]

[SL II] R. P. Langlands and D. Shelstad, *Orbital integrals on forms of* SL(3), II, Can. J. Math., **XLI**, (1989) 480–507. [doi:10.4153/CJM-1989-022-0.]

[PV] R. P. Langlands and D. Shelstad, *On principal values on p-adic manifolds*, in Lie Group Representations, II, Spring Lecture Notes, vol. 1041, 1984. [doi:10.1007/BFb0073150.]

[TF I] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann., **278** (1987), 219–271. [doi:10.1007/bf01458070.]

[TF II] R. P. Langlands and D. Shelstad, *Descent for transfer factors*, The Grothendieck Festschrift **II** (1990), 485–563. [doi:10.1007/978-0-8176-4575-5_12.]

[R] J. Rogawski, *Automorphic representations of unitary groups in three variables*, Ann. of. Math. Study **123** (1990).

[S] Diana Shelstad, *L-Indistinguishability for real groups*, Math. Ann., **259**, (1982), 385–430.

[W] D. V. Widder, *The Laplace Transform*, Princeton University Press, 1946.

Compiled on May 1, 2026.