A SCHOTTKY-LANDAU THEOREM FOR HOLOMORPHIC MAPPINGS IN SEVERAL COMPLEX VARIABLES (*)

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1. Statement of the main theorem.

Let V be a smooth projective variety of complex dimension n. Denote by $B[r] = \{z \in C^n : \|z\| < r\}$ the ball of radius r in C^n. We will study holomorphic mappings

(1.1) $$f: B[r] \to V - D$$

where D is a divisor on V and which are *non-degenerate* in the sense that the image $f(B[r])$ contains an open set in V. Fixing a point $w_0 \in V - D$ and local coordinates $(w_1, \ldots, w_n)$ around w_0, we let $\mathcal{F}$ be the class of all mappings which are normalized by the conditions

(1.2) $$\begin{cases} f(0) = w_0 \\ \left| \det \left(\dfrac{\partial w_j(z)}{\partial z_k} \right)_{z=0} \right| = 1 . \end{cases}$$

Given $f \in \mathcal{F}$, we denote by $\varrho(f)$ the supremum of all r such that f is defined on $B[r]$.

(1.3) DEFINITION: The pair (V, D) satisfies the *Schottky-Landau property* if, for every $w_0 \in V - D$, there exists a constant $r_0 = r_0(w_0, V, D)$ such that $\varrho(f) \leq r_0$ for every $f \in \mathcal{F}$.

If (V, D) satisfies the Schottky-Landau property, then obviously every entire map

$$f: C^n \to V - D$$

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is degenerate (*little Picard theorem*). It is of course a classical theorem that $(\boldsymbol{P}_1, \{0, 1, \infty\})$ satisfies the Schottky-Landau property. Our problem is to generalize this to the case of several variables. Roughly speaking we will find that (V, D) satisfies the Schottky-Landau property if

$$D + K_V > \operatorname{sing}(D) \tag{1.4}$$

where K_V is the canonical divisor on V and $\operatorname{sing}(D)$ is a non-negative quantity which measures how much more singular D is than having only normal crossings.

To make this precise, we will say that a divisor D on V has *ordinary singularities* if globally

$$D = D_1 + \ldots + D_l \tag{1.5}$$

where the D_j are non-singular hypersurfaces which are nowhere tangent. The divisor D has *normal crossings* if it has ordinary singularities and if locally the tangent spaces to the irreducible components D_j are linearly independent.

EXAMPLE: Let $V = \boldsymbol{P}_n$ and $D = H_1 + \ldots + H_N$ be a linear combination of hyperplanes. Then D has ordinary singularities if, and only if, the H_j are *distinct*; and D has normal crossings if, and only if, the H_j are in *general position* in the sense that any $k \leq n+1$ of the H_j are linearly independent.

Let D have ordinary singularities. To each $z \in D$ we want to assign an integer $\mu(z)$ which will measure the deviation from normal crossings at z. Let D be given around z by (1.5), and suppose that k of the tangent spaces at z are linearly independent. Then we may choose coordinates $(z_1, \ldots, z_k; z_{k+1}, \ldots, z_n)$ around z such that all of the local branches $D_1, \ldots, D_l$ are given by

$$\left\{\begin{array}{l} z_1 = 0\,; \\ \vdots \\ z_k = 0\,; \\ L_1(z_1, \ldots, z_k) + [2] = 0\,; \\ \vdots \\ L_{l-k}(z_1, \ldots, z_k) + [2] = 0\,; \end{array}\right. \tag{1.6}$$

where the $L_\alpha(z_1, \ldots, z_k)$ are distinct linear forms and [2] denotes higher order terms. We define the *exceptional multiplicity* $\mu(z) = l - k$. The *exceptionally singular locus* S will be the subvariety where $\mu(z) > 0$.

Obviously we may write $S = S_1 + \ldots + S_M$ where $\mu(z)$ is a constant $\mu(S_\alpha)$ on each S_α. From (1.6) it follows that S_α is smooth and

$$\mu(S_\alpha) + \operatorname{codim}(S_\alpha) = \operatorname{mult}(D_1 + \ldots + D_l)\,. \tag{1.7}$$

EXAMPLE: If V is the projective plane $\boldsymbol{P}_2$ and $D = L_1 + \ldots + L_5$ is a sum of five lines, three of which pass through a point p

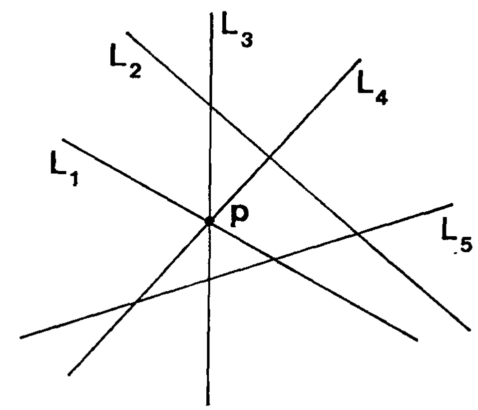

Figure 1

then S is that point and $\mu(S) = 1$.

We let $[D]$ be the line bundle corresponding to the divisor D and $c_1([D])$ be the *Chern class* (cf. [3], section 1(*a*)) of this line bundle. Then $c_1([D])$ is a real (1, 1) form on V whose magnitude measures how ample D is. Suppose that D has ordinary singularities and let S be its exceptionally singular locus. We want to define a (1,1) form $c(S)$ which will measure how large S is. Using the formula

$$c(S) = \mu(S_1)c(S_1) + \ldots + \mu(S_M)c(S_M)\,,$$

it will suffice to define the individual terms $c(S_\alpha)$. We will define $c(S_\alpha)$ under the assumption that there is a holomorphic vector bundle $E_\alpha \to V$ and holomorphic section $\sigma_\alpha \in H^0(V, \mathcal{O}(E_\alpha))$ such that S_α is given by $\sigma_\alpha = 0$. This will suffice for our applications, and the general case can be done by a similar method. Choose an Hermitian metric in $E_\alpha \to V$ and let ϑ_α be the curvature of the associated connection (cf. [5], page 194). Then

$$c(S_\alpha) = \left(\frac{\sqrt{-1}}{2\pi}\right)\frac{(\vartheta_\alpha \sigma_\alpha, \sigma_\alpha)}{(\sigma_\alpha, \sigma_\alpha)}\,. \tag{1.8}$$

EXAMPLE: Suppose that $V = \boldsymbol{P}_n$ and ω is the standard Kähler form. If $D = H_1 + \ldots + H_N$ consists of distinct hyperplanes, then each component of the exceptional singular locus will be a linear subspace of $\boldsymbol{P}_n$. It follows that

$$c(S) = \left\{\sum_{\alpha=1}^{M} \mu(S_\alpha)\right\}\omega \tag{1.9}$$

is the sum of the exceptional multiplicities of the components of S times the form ω.

Let $K_V \to V$ be the canonical line bundle of V. The main result of this paper is

(1.10) THEOREM: The pair (V, D) satisfies the Schottky-Landau property if D has ordinary singularities and if

$$c_1([D]) + c_1(K_V) > c(S)$$

where S is the exceptionally singular locus of D.

This theorem was given in [3] in case D has only normal crossings. Our proof here is more direct than that in [3], and is based on the construction of a volume form in the next section. For normal crossings, the main theorem is sharp (take $V = \boldsymbol{P}_n$ and $D = H_1 + \ldots + H_{n+1}$ a linear combination of $n+1$ hyperplanes in general position). Our main theorem says that, as D acquires worse singularities, then its Chern class must be made larger to compensate for these. This is geometrically quite reasonable, and some examples discussed in § 4 indicate this result may also be sharp.

2. Construction of a volume form.

Let V be a smooth variety and $D \subset V$ a divisor with ordinary singularities. We refer to § 1(*a*) of [3] for the definition of a *volume form* and its associated *Ricci form* on a complex manifold.

(2.1) PROPOSITION: Under the assumption

$$c_1([D]) + c_1(K_V) > c(S) ,$$

we may find a volume form Ψ on $V - D$ such that

$$\begin{cases} \Psi < (\operatorname{Ric}\Psi)^n \\ \int\limits_{V-D} (\operatorname{Ric}\Psi)^n < \infty . \end{cases}$$

The proof proceeds in two steps.

Step one: the case of normal crossings. Let Ω be a C^∞ volume form on V. Then (cf. (1.5) in [3]) $\operatorname{Ric}(\Omega) = c_1(K_V)$. We may find a metric in the line bundle $[D] \to V$ such that the length $|\sigma|$ of the canonical section $\sigma \in H^0(V, \mathcal{O}([D]))$ satisfies (cf. (1.7) in [3])

$$- dd^c \log |\sigma|^2 = c_1([D]) . \tag{2.2}$$

Multiplying σ by a suitable constant, we may assume that

$$|\sigma| < \delta$$

for any given $\delta > 0$. We will show that, by choosing ε and δ sufficiently small, the volume form

$$\Psi = \frac{\Omega}{(\log |\sigma|^2)^2 |\sigma|^{2-2\varepsilon}} \tag{2.3}$$

will satisfy the requirements of our proposition. From (2.2) and (2.3) we have

$$\operatorname{Ric}\Psi = c_1(K_V) + (1-\varepsilon)c_1([D]) - dd^c \log(\log|\sigma|^2)^2 . \tag{2.4}$$

Choose ε sufficiently small so that the C^∞ (1, 1) form

$$\omega_1 = (1-\varepsilon)c_1([D]) + c_1(K_V)$$

is positive on all of V. Now

$$- dd^c \log(\log|\sigma|^2)^2 = -\frac{dd^c \log|\sigma|^2}{\log|\sigma|^2} + \frac{\sqrt{-1}}{2\pi}\left\{\frac{\partial \log|\sigma|^2 \wedge \bar\partial \log|\sigma|^2}{(\log|\sigma|^2)^2}\right\}. \tag{2.5}$$

Since $dd^c \log|\sigma|^2$ is C^∞ on V, by making δ sufficiently small we may absorb the $-dd^c\log|\sigma|^2/\log|\sigma|^2$ term into ω_1. Thus

$$\text{(2.6)} \qquad \operatorname{Ric}\Psi = \omega_2 + \frac{\sqrt{-1}}{2\pi}\left\{\frac{\partial\log|\sigma|^2\wedge\bar\partial\log|\sigma|^2}{(\log|\sigma|^2)^2}\right\}$$

where ω_2 is positive and C^∞ on V. Since the remaining term on the R.H.S. of (2.6) is non-negative, we may localize around a point $z_0 \in D$. Because D is assumed to have normal crossings, we may choose local holomorphic coordinates $z_1, \ldots, z_n$ such that D is given by

$$\zeta = z_1 \ldots z_k = 0\ .$$

We will have an inequality

$$\text{(2.7)} \qquad \operatorname{Ric}\Psi \geqq \frac{\alpha\sqrt{-1}}{2\pi}\left\{\sum_{j=1}^{n} dz_j\wedge d\bar z_j\right\} + \frac{\sqrt{-1}}{2\pi}\left\{\frac{\partial\log|\sigma|^2\wedge\bar\partial\log|\sigma|^2}{(\log|\sigma|^2)^2}\right\}$$

for some $\alpha > 0$. By a change of scale, we may assume that $\alpha = 1$. Now $|\sigma|^2 = a\zeta\bar\zeta$ for some positive C^∞ function a. Setting

$$\vartheta_\mu = \frac{dz_\mu}{|\log|\zeta|^2|\cdot|z_\mu|}\,,$$

the second term on the R.H.S. of (2.7) is of the form

$$\text{(2.8)} \qquad \frac{\omega}{(\log|\zeta|^2)^2} + \sum_{\mu=1}^{k}\frac{\lambda_\mu}{|\log|\zeta|^2|}\wedge\vartheta_\mu + \frac{\sqrt{-1}}{2\pi}\sum_{\nu=1}^{k}\vartheta_\nu\wedge\bar\vartheta_\nu$$

where ω and the λ_μ are bounded. We may absorb the first term in (2.8) as before, and then (2.8) gives an inequality

$$\text{(2.9)} \qquad \operatorname{Ric}\Psi \geqq \frac{\sqrt{-1}}{2\pi}\left\{\sum_{j=1}^{n} dz_j\wedge d\bar z_j\right\} + \sum_{\mu=1}^{k}\frac{\lambda_\mu\wedge\vartheta_\mu}{|\log|\zeta|^2|} + \frac{\sqrt{-1}}{2\pi}\sum_{\nu=1}^{k}\vartheta_\nu\wedge\bar\vartheta_\nu\,.$$

Applying the Cauchy-Schwarz inequality in (2.9) and perhaps shrinking δ gives

$$\text{(2.10)} \qquad \operatorname{Ric}\Psi \geqq \frac{\beta\sqrt{-1}}{2\pi}\left\{\sum_{j=1}^{n} dz_j\wedge d\bar z_j + \sum_{\nu=1}^{k}\vartheta_\nu\wedge\bar\vartheta_\nu\right\}$$

for some $\beta > 0$. From (2.10) we find

$$\text{(2.11)} \qquad (\operatorname{Ric}\Psi)^n \geqq \frac{\beta^n}{(\log|\zeta|^2)^{2k}|\zeta|^2}\left\{\prod_{j=1}^{n}\left(\frac{\sqrt{-1}}{2\pi}\,dz_j\wedge d\bar z_j\right)\right\}.$$

Using this together with the compactness of D it follows that $(\operatorname{Ric}\Psi)^n \geqq c\Psi$ for some positive constant c. Since $\operatorname{Ric}(c\Psi) = \operatorname{Ric}(\Psi)$, we may make a change of scale to have $(\operatorname{Ric}\Psi)^n \geqq \Psi$.

To verify that the integral of $(\operatorname{Ric}\Psi)^n$ is finite, we have locally the easy upper bound

$$\text{(2.12)} \qquad (\operatorname{Ric}\Psi)^n \leqq \frac{c'}{(\log|\zeta|^2)^{2k}|\zeta|^2}\prod_{j=1}^{n}\left(\frac{\sqrt{-1}}{2}\,dz_j\wedge d\bar z_j\right)$$

for some $c' > 0$. The inequality of arithmetic and geometric means gives

$$(\log|\zeta|^2)^k \geqq (\log|z_1|^2)^2 \ldots (\log|z_k|^2)^2\ .$$

Combining this with (2.12) we obtain

$$(\operatorname{Ric}\Psi)^n \leqq c''\prod_{\nu=1}^{k}\left(\frac{\sqrt{-1}}{2}\,\frac{dz_\nu\wedge d\bar z_\nu}{(\log|z_\nu|^2)^2|z_\nu|^2}\right)\prod_{\alpha=k+1}^{n}\left(\frac{\sqrt{-1}}{2}\,dz_\alpha\wedge d\bar z_\alpha\right),$$

and the finiteness of $\int_{V-D}(\operatorname{Ric}\Psi)^n$ follows from this and $\int_0^\varepsilon dr/(\log r^2)^2 r < \infty$.

(2.13) Remark: With a little more work, we could show the existence of a volume Ψ satisfying

$$\text{(2.14)} \qquad \begin{cases} \Psi \leqq \operatorname{Ric}(\Psi)^n \leqq c\Psi & (c > 0) \\ \int_{V-D}(\operatorname{Ric}\Psi)^n < \infty & \end{cases}$$

and such that locally

$$\text{(2.15)} \qquad \Psi = a(z)\prod_{\nu=1}^{k}\left(\frac{\sqrt{-1}}{2\pi}\,\frac{dz_\nu\wedge d\bar z_\nu}{|z_\nu|^2(\log|z_\nu|^2)^2}\right)\prod_{\alpha=k+1}^{n}\left(\frac{\sqrt{-1}}{2\pi}\,dz_\alpha\wedge d\bar z_\alpha\right)$$

where $a(z)$ is a positive C^∞ function. The geometric meaning of (2.15) will be given in §4(b) below.

Step two: reduction to the case of normal crossings. Let D have ordinary singularities with exceptional singular locus S. We will discuss the case when S is irreducible, as the general argument is the same. Suppose thas codim $(S) = k$ (cf. (1.6)) and let $\mu = \mu(S)$ be the exceptional multiplicity of S. Then

$$\mu + k = \nu \tag{2.16}$$

where ν is the actual multiplicity of D (cf. (1.7)). Let

$$\tilde{V} \xrightarrow{\pi} V$$

be the quadratic transform of V along S. We denote by $\tilde{E} = \pi^{-1}(S)$ the inverse image of S under π and $\tilde{D}$ the proper transform of D. Then

$$V - D = \tilde{V} - (\tilde{D} + \tilde{E}), \tag{2.17}$$

and we have the formulae

$$\begin{cases} \pi^*[D] = [\tilde{D} + \nu\tilde{E}] \\ K_{\tilde{V}} = \pi^* K_V \oplus [\tilde{E}]^{(k-1)} . \end{cases} \tag{2.18}$$

Combining (2.16) and (2.18) it follows that

$$c_1([\tilde{D} + \tilde{E}]) + c_1(K_{\tilde{V}}) = \pi^*\{c_1([D]) + c_1(K_V)\} - \mu c_1([\tilde{E}]) . \tag{2.19}$$

To prove our proposition, it will suffice to show that the R.H.S. of (2.19) is everywhere positive on $\tilde{V}$. Let $E \to V$ be postulated the holomorphic vector bundle with section σ such that S is given $\sigma = 0$. Then there is a natural inclusion

$$[\tilde{E}] \hookrightarrow \pi^* E$$

such that the length of the canonical section $e \in H^0(\tilde{V}, \mathcal{O}([\tilde{E}]))$ is given by

$$|e|^2 = (\sigma, \sigma) .$$

It follows that (cf. (1.7) in [3])

$$-c_1([\tilde{E}]) = dd^c \log |e|^2 = dd^c \log (\sigma, \sigma) .$$

On the other hand, from (2.44) in [5] we have

$$dd^c \log (\sigma, \sigma) = \frac{\sqrt{-1}}{2\pi}\left\{\frac{(D'\sigma, D'\sigma) - (D'\sigma, \sigma)(\sigma, D'\sigma)}{(\sigma, \sigma)^2}\right\} - \frac{\sqrt{-1}}{2\pi}\frac{(\vartheta\sigma, \sigma)}{(\sigma, \sigma)} . \tag{2.20}$$

The (1, 1) form

$$\frac{\sqrt{-1}}{2\pi}\left\{\frac{(D'\sigma, D'\sigma) - (D'\sigma, \sigma)(\sigma, D'\sigma)}{(\sigma, \sigma)^2}\right\}$$

is everywhere non-negative on $\tilde{V}$ and is positive definite along the fibres of

$$\pi: \tilde{E} \to S$$

(cf. page 202 in [5]). On the other hand, we have assumed that

$$c_1([D]) + c_1(K_V) - \frac{\mu\sqrt{-1}}{2\pi}\frac{(\vartheta\sigma, \sigma)}{(\sigma, \sigma)} > 0$$

on V. It follows then that the R.H.S. of (2.19) is everywhere positive on $\tilde{V}$. Q.E.D.

3. Proof of the Schottky-Landau theorem.

Let M be a complex manifold of dimension n and on which there is a volume form Ψ satisfying

$$(\mathrm{Ric}\,\Psi)^n \geqq \Psi . \tag{3.1}$$

Fixing a point $x_0 \in M$, we let $\mathcal{F}$ be the class of all holomorphic mappings $f: B[r] \to M$ which satisfy the normalization conditions

$$\begin{cases} f(0) = x_0 \\ (f^*\Psi)(0) \geqq \prod_{j=1}^{n}\left(\frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j\right). \end{cases} \tag{3.2}$$

Given $f \in \mathcal{F}$ we denote by $\varrho(f)$ the supremum of all r such that f is defined on $B[r]$.

(3.3) PROPOSITION: There exists a constant $r_0 = r_0(M, x_0)$ such that $\varrho(f) \leqq r_0$ for all $f \in \mathcal{F}$.

This result follows directly from known theorems of Ahlfors, Chern, and Kobayashi (cf. [1] and [7]). For completeness we shall give a proof here.

Let $\Psi[r]$ be the unique volume form on $B[r]$ which is invariant under all holomorphic automorphisms of the ball and which satisfies

(3.4) $$(\mathrm{Ric}\,\Psi[r])^n = \Psi[r].$$

On the unit ball $B[1]$, there is the well-known explicit formula

(3.5) $$\Psi[1] = \frac{c}{(1-\|z\|^2)^{n+1}} \prod_{j=1}^{n} \left(\frac{\sqrt{-1}}{2\pi}\, dz_j \wedge d\bar{z}_j\right)$$

for a suitable constant $c > 0$. The general form $\Psi[r]$ is obtained from $\Psi[1]$ by dilation, and, in particular,

(3.6) $$\Psi[r](0) = \frac{c \prod_{j=1}^{n} \left((\sqrt{-1}/2\pi)\, dz_j \wedge d\bar{z}_j\right)}{r^{2n}}.$$

(3.7) LEMMA (*Ahlfors, Chern, Kobayashi*): For $f \in \mathcal{F}$ we have the volume-decreasing estimate

$$f^*\Psi \leqq \Psi[r].$$

Assuming this lemma, we obtain from (3.6) and (3.2) that

$$\Phi \leqq f^*\Psi(0) \leqq \frac{c\Phi}{r^{2n}}$$

where $\Phi = \prod_{j=1}^{n} \left((\sqrt{-1}/2\pi)\, dz_j \wedge d\bar{z}_j\right)$. Thus

$$r \leqq \sqrt[2n]{c},$$

so that we may take $r = \sqrt[2n]{c}$ to obtain our proposition.

PROOF OF LEMMA (3.7): For $\varrho \leqq r$ we write on $B[\varrho]$

$$f^*\Psi = \xi_\varrho \cdot \Psi[\varrho].$$

Then, from (3.5), it follows that

(3.8) $$\lim_{\varrho \to r} \xi_\varrho(z) = \xi_r(z) \qquad (z \in B[r]).$$

On the other hand, for $\varrho < r$ we have again from (3.5) that

(3.9) $$\lim_{\|z\| \to \varrho} \xi_\varrho(z) = 0.$$

Combing (3.8) and (3.9), we see that the estimate $\xi_r \leqq 1$ will follow from $\xi_\varrho \leqq 1$ for all $\varrho < 1$, while ξ_ϱ obviously attains an interior maximum point in $B[\varrho]$. Thus it will suffice to assume that $\xi_r(z)$ has a maximun at $z_0 \in B[r]$. But then, from (1.5) in [3],

(3.10) $$0 \geqq dd^c \log \xi_r(z_0) = f^*(\mathrm{Ric}\,\Psi)(z_0) - \mathrm{Ric}\,(\Psi[r])(z_0).$$

Taking n-th exterior powers in (3.10) and using (3.1) and (3.4) gives

$$f^*\Psi(z_0) \leqq \Psi[r](z_0),$$

or equivalently $\xi_r(z_0) \leqq 1$, which is what we wanted to prove. Q.E.D.

4. Applications, comments and references.

a) *The case of polycylinders*. The volume form $4\sqrt{-1} \cdot \{r^2 dz \wedge d\bar{z})/(r^2 - |z|^2)^2\}$ on the disc $|z| < r$ induces a volume form $\vartheta_{P[r]}$ on the polycylinder $P[r] = \{z \in \mathbf{C}^n : |z_j| < r_j\}$. This *Poincaré volume form* $\vartheta_{P[r]}$ has the property

(4.1) $$(\mathrm{Ric}\,\vartheta_{P[r]})^n = \vartheta_{P[r]},$$

and the proof of our main theorem applies to the case of polycylinders to give the

(4.2) PROPOSITION: Let (V, D) and $\mathcal{F}$ be as in the statement of Theorem (1.10). Then, for any $f \in \mathcal{F}$ the maximal polycylinder $P[r]$ on which f is defined satisfies the inequality

$$r_1 \ldots r_n \leqq r_0$$

for some constant r_0.

Recall that a *punctured polycylinder* P^* is given by $\{z \in \mathbf{C}^n : |z_j| < 1, z_1 \dots z_k \neq 0\}$. Using the universal covering $P[1] \to P^*$, the Poincaré volume form on $P[1]$ induces a volume form ϑ_{P^*} on P^*. Explicitly

$$\vartheta_{P^*} = c \prod_{\mu=1}^{k} \left(\frac{\sqrt{-1}\, dz_\mu \wedge d\bar{z}_\mu}{(\log |z_\mu|^2)^2 |z_\mu|^2} \right) \prod_{\alpha=k+1}^{n} (\sqrt{-1}\, dz_\alpha \wedge d\bar{z}_\alpha) \tag{4.3}$$

for a suitable positive constant $c > 0$. Lemma (3.7) adapted to the present context gives

(4.4) LEMMA: Let Ψ be a volume form on P^* which satisfies $(\mathrm{Ric}\, \Psi)^n \geq \Psi$. Then

$$\begin{cases} \Psi \leq \vartheta_{P^*} \\ \displaystyle\int_{V-D} \Psi < \infty . \end{cases}$$

REMARKS: The use of volume forms in the study of holomorphic mappings was begun by Chern [1] and Kobayashi [7]. Applications of volume forms to holomorphic mappings into algebraic varieties are given in [2], [3], and [6]. Lemma 4.4 should provide motivation for the expression for Ψ given by (2.3).

Remark (2.13) together with (2.15) yield a proposition which will be used at a later time to derive defect relations for entire holomorphic mappings. To state this, we let V be a smooth projective variety, $D \subset V$ a divisor with ordinary singularities and exceptional singular locus S, and

$$\begin{array}{ccc} \tilde{D} & \hookrightarrow & \tilde{V} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ D & \hookrightarrow & V \end{array}$$

the standard desingularization of D. We will use the notation

$$\Psi \sim \Phi$$

to mean that two volume forms Ψ, Φ satisfy

$$c_1 \Psi \leq \Phi \leq c_2 \Psi \qquad (c_1, c_2 > 0) .$$

(4.5) PROPOSITION: If $c_1([D]) + c_1(K_V) > c(S)$, then there exists a volume form Ψ on $V - D$ satisfying

$$\begin{cases} (\mathrm{Ric}\, \Psi)^n \sim \Psi \\ \displaystyle\int_{V-D} \Psi < \infty . \end{cases}$$

Moreover, if $P^* \subset \tilde{V} - \tilde{D}$ is a punctured polycylinder around some branches of $\tilde{D}$, then

$$\pi^*(\Psi) | P^* \sim \vartheta_{P^*}$$

where the Poincaré volume form ϑ_{P^*} is given by (4.3).

b) *Linear spaces in* $\mathbf{P}_n$. We want to see how sharp our results are, and for this we will examine linear hyperplanes in $\mathbf{P}_n$. Let $D = H_1 + \dots + H_N$ be a linear combination of distinct hyperplanes in $\mathbf{P}_n$. Thus D is a divisor with ordinary singularities, and we have the

(4.6) LEMMA: The inequality $c_1([D]) + c_1(K_{\mathbf{P}_n}) > c(S)$ holds if, and only if, there are at least $n+2$ hyperplanes in general position among the H_μ.

PROOF: We identify $H^2(\mathbf{P}_n, \mathbf{Z})$ with $\mathbf{Z}$ in the usual way, so that in particular

$$\begin{cases} c_1([H_\mu]) = 1 \\ c_1(K_{\mathbf{P}_n}) = -(n+1) . \end{cases} \tag{4.7}$$

Let $D_\mu = H_1 + \dots + H_\mu$ and suppose that S_μ is the exceptionally singular locus of D_μ. Using the notation

$$\delta(\mu) = c_1([D_\mu]) + c_1(K_{\mathbf{P}_n}) - c(S_\mu) ,$$

it follows from (1.9) and (4.7) that

$$\begin{cases} \delta(\mu) \leq \delta(\mu+1) \\ \delta(\mu) + 1 \geq \delta(\mu+1) . \end{cases}$$

Moreover, it follows from (1.9) that $\delta(\mu+1) = \delta(\mu) + 1$ if, and only if, $H_{\mu+1}$ is in general position with respect to a maximal subset of $\{H_1, \dots, H_\mu\}$ which is itself in general position. The lemma is a consequence of this together with (4.7). Q.E.D.