SHIMURA VARIETIES AT INFINITE PLACES

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If a reductive group G over $\mathbf{Q}$ and a homomorphism $h : S \to G$ over $\mathbf{R}$ which satisfy the axioms of *Travaux de Shimura* are given, then a weakly canonical model $\{S_K/E\}$ of the family of Shimura varieties

$$S_K(\mathbf{C}) = G(\mathbf{Q})\backslash G(\mathbf{A})/K_\infty K$$

must by definition be given over a subfield E of $\mathbf{C}$. Two problems suggest themselves.

One suspects immediately and the suspicion is strongly reinforced by Kazhdan's paper *Arithmetic varieties and their fields of quasi-definition* that if τ is an automorphism of $\mathbf{C}$ then $\{{}^\tau S_K/{}^\tau E\}$ is again a weakly canonical model for the family of Shimura varieties associated to some pair $\{{}^\tau G, {}^\tau h\}$. The first problem is to describe $\{{}^\tau G, {}^\tau h\}$ in terms of G, h, and τ.

If E is real and τ is the complex conjugation then τ defines an involution on each of the manifolds $S_K(\mathbf{C})$. the second problem is to describe this involution in terms of G and h. I shall describe below the form I believe the solution to these two problems will take. One might hope to prove that this form is correct for those families which can be treated by the methods of Shimura and Deligne. In attempting to do so, I have been led to further problems. At the moment these problems remain quite vague; I have not even been able to guess the exact form their solutions might take.

The second problem has already been broached by Shimura and Shih. Unlike them, I shall work with connected groups and use a strictly adelic formulation. This is just a matter of temperament; I do not believe it detracts substantially from the generality.

The group ${}^\tau G$ will it be obtained from G by an inner twisting. This twisting will be trivial except at infinity, so that

$${}^\tau G(\mathbf{A}_f) = G(\mathbf{A}_f).$$

K may therefore be regarded as a subgroup of ${}^\tau G(\mathbf{A}_f)$.

If T is a torus in G let T_{ad} be its image in G_{ad}. I take T to be defined over $\mathbf{Q}$ and such that $T_{\text{ad}}(\mathbf{R})$ is compact. Composing h with $\operatorname{ad} x$, $x \in G(\mathbf{R})$, if necessary, we may suppose that

$$h : S \to T.$$

Then composing

$$\begin{array}{c} \mathrm{GL}(1) \\ \big\downarrow \\ \mathrm{GL}(1) \times \mathrm{GL}(1) \longrightarrow S \end{array}$$

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with h we obtain a coweight $\widehat{\mu}$ of T. If $\widehat{L}(T_{\mathrm{ad}})$ is the lattice of coweights of T_{ad} then

$$\widehat{L}(T_{\mathrm{ad}})/2\widehat{L}(T_{\mathrm{ad}}) \simeq H^{-1}\Big(\mathfrak{G}(\mathbf{C}/\mathbf{R}), \widehat{L}(T_{\mathrm{ad}})\Big) \simeq H^{1}\big(\mathfrak{G}(\mathbf{C}/\mathbf{R}), T_{\mathrm{ad}}(\mathbf{C})\big).$$

The image of $\tau\widehat{\mu} - \widehat{\mu}$ in the group on the left defines an element in the group on the right. This gives the twisting cocycle at infinity.

This twisting does not depend on any of the choices made. To see this one combines some simple fiddling around with 1-cocycles with the observation that if ω lies in the Weyl group then $\omega\widehat{\mu} - \widehat{\mu}$ gives a trivial twisting. If ω is the image of w in the normalizer of T in G the corresponding 1-cocycle arises from the false twisting

$$T \xrightarrow{w} T \qquad G \xrightarrow{w} G.$$

Take F to be a large finite Galois extension of $\mathbf{Q}$. The bottom row of the diagram

$$\begin{array}{ccccc} & & H^{1}\big(\mathfrak{G}(\mathbf{C}/\mathbf{R}), T_{\mathrm{ad}}(\mathbf{C})\big) & & \\ & & \downarrow & \searrow & \\ H^{1}\big(\mathfrak{G}(F/\mathbf{Q}), T_{\mathrm{ad}}(F)\big) & \to & H^{1}\big(\mathfrak{G}(F/\mathbf{Q}), T_{\mathrm{ad}}(\mathbf{A}_F)\big) & \to & H^{1}\Big(\mathfrak{G}(F/\mathbf{Q}), \widehat{L}(T_{\mathrm{ad}}) \otimes C_F\Big) \end{array}$$

is exact. Moreover the diagram

$$\begin{array}{ccc} H^{1}\big(\mathfrak{G}(\mathbf{C}/\mathbf{R}), T_{\mathrm{ad}}(\mathbf{C})\big) & \longrightarrow & H^{1}\Big(\mathfrak{G}(F/\mathbf{Q}), \widehat{L}(T_{\mathrm{ad}}) \otimes C_F\Big) \\ \uparrow & & \uparrow \\ H^{-1}\Big(\mathfrak{G}(\mathbf{C}/\mathbf{R}), \widehat{L}(T_{\mathrm{ad}})\Big) & \longrightarrow & H^{-1}\Big(\mathfrak{G}(F/\mathbf{Q}), \widehat{L}(T_{\mathrm{ad}})\Big) \end{array}$$

in which the bottom arrow is given by $\widehat{\lambda} \to \widehat{\lambda}$ is commutative. Since $\tau\widehat{\mu} - \widehat{\mu}$ clearly has image 0 in $H^{-1}\Big(\mathfrak{G}(F/\mathbf{Q}), \widehat{L}(T_{\mathrm{ad}})\Big)$, there clearly is an element of $H^{1}\big(\mathfrak{G}(F/\mathbf{Q}), T_{\mathrm{ad}}(F)\big)$ which is trivial at every finite place and which is the given 1-cocycle at the infinite place. This gives us the global twisting. By Hasse's principle, it is well-defined.

${}^{\tau}G$ comes equipped with $T \to {}^{\tau}G$. I now define ${}^{\tau}h$. Let ρ be complex conjugation. Then ${}^{\tau}h$ is the composition

$$S \simeq \mathrm{GL}(1) \times \mathrm{GL}(1) \xrightarrow{(\tau\widehat{\mu}, \rho\tau\widehat{\mu})} T \longrightarrow {}^{\tau}G \quad .$$

The roots of T can be classified as compact or non-compact with respect to G or with respect to ${}^{\tau}G$. The twisting changes the type of α if and only if

$$(-1)^{\langle \alpha, \tau\widehat{\mu} - \widehat{\mu}\rangle} = -1.$$

Thus α is compact or non-compact with respect to ${}^{\tau}G$ according as $(-1)^{\langle \alpha, \tau\widehat{\mu}\rangle}$ is 1 or -1. It follows that $({}^{\tau}G, {}^{\tau}h)$ satisfies the axioms of Deligne.

Now suppose E is real and τ is complex conjugation. Then, if h is taken to factor through T, there is an ω in the Weyl group of T so that

$$\tau\widehat{\mu} = \omega\widehat{\mu}.$$

However $\tau\widehat{\mu} = -\widehat{\mu}$. Thus ω must take roots of type $(0, 0)$ to roots of the same type and roots of type $(1, -1)$ to roots of type $(-1, 1)$. (I use here the notation of Hodge structures, favoured by Deligne.) I shall show in a moment that ω has a representative w in $G(\mathbf{R})$. This element w normalizes K_∞ and as uniquely determined modulo K_∞. Thus the map $g \to gw$ on $G(\mathbf{A})$ yields upon passage to quotients a well-defined homomorphism of

$$S_K(\mathbf{C}) = G(\mathbf{Q})\backslash G(\mathbf{A})/K_\infty K.$$

This should be the involution.

To establish the existence of w, I use the results and notation of Harish-Chandra's *Representations of Semi-Simple Lie Groups IV* §6. He introduces a set $\{\gamma_1, \dots, \gamma_s\}$ of non-compact roots and associates to each γ_i a homomorphism

$$\varphi_i : \mathrm{SL}(2) \to G$$

which is defined over $\mathbf{R}$. Set

$$w_i = \varphi_i\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right)$$

The w_i commute. Tentatively take

$$w_0 = \prod_{i=1}^{s} w_i.$$

w_0 lies in $G(\mathbf{C})$. However w_0 does normalize T and takes γ_i to $-\gamma_i$. It therefore takes positive non-compact roots to negative non-compact roots and its image in the Weyl group is ω.

Since the roots γ_i are orthogonal

$$\left\langle \gamma_j, \sum_i \widehat{\gamma}_i \right\rangle = 2.$$

Here $\widehat{\gamma}_i$ is a coroot. Moreover

$$\langle \gamma_j, \widehat{\mu} - \rho\widehat{\mu} \rangle - 2\langle \gamma_j, \widehat{\mu} \rangle = 2.$$

Since $\rho\widehat{\mu} - \widehat{\mu}$ is 0 on the set of fixed points of w_0, it is in particular 0 on the orthogonal complement of $\{\gamma_1, \dots, \gamma_s\}$. Thus

$$\sum_i \widehat{\gamma}_i = \widehat{\mu} - \rho\widehat{\mu}.$$

However if λ is a weight,

$$\lambda\Big(\rho(w_0)w_0^{-1}\Big) = \lambda\left(\prod \varphi_i\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)\right) = (-1)^{\langle \lambda, \sum \widehat{\gamma}_i \rangle}.$$

Let s in T be defined by

$$\lambda(s) = (-1)^{\langle \lambda, \widehat{\mu} \rangle}.$$

Then

$$\lambda\big(\rho(s)\big) = (-1)^{\langle \lambda, \rho\widehat{\mu} \rangle}$$

and if $w = sw_0$ then

$$\lambda\Big(\rho(w)w^{-1}\Big) = 1.$$

Consequently $w \in G(\mathbf{R})$.

The technique of construction exposed by Deligne involves five steps. To verify a statement for the groups amenable to this treatment one must:

(i) Verify it for abelian groups.
(ii) Verify it for the group of symplectic similitudes.

(iii) Verify that if we have $(G_1, h_1) \hookrightarrow (G_2, h_2)$ and if the statement is true for the larger group than it is true for the smaller.
(iv) Verify that if it is true for (G, h) and $\delta : S \to C$ where C is the connected component of the centre of G, then it is also true for $(G, h\delta)$.
(v) Devise a method of descent.

The solution I have suggested to the second problem, is unsuitable for descent and the solution for the second is unsuitable for passage to a subgroup; so a reformulation is necessary.

To clarify the situation with respect to the second problem, I shall prove a lemma, otherwise rather useless. Let Z be the centre of G and let Z^0 be its connected component in the algebraic sense. Set

$$S(\mathbf{C}) = \varprojlim_K S(\mathbf{C}).$$

If $E(G, h) \subseteq E \subseteq \mathbf{R}$ let

$$\rho : S(\mathbf{C}) \to S(\mathbf{C})$$

be the involution defined by complex conjugation and let φ be the map obtained from $g \to gw$. Then $\psi = \rho \circ \varphi^{-1}$ is complex analytic on each $S_K(\mathbf{C})$ and commutes with the action of $G(\mathbf{A}_f)$. To see the possibilities for ρ we need only see those for ψ.

Suppose $x \in g(\mathbf{A}_f)$ and x normalizes $G(\mathbf{Q})$. Set

$$\gamma' = x\gamma x^{-1}.$$

If $x = (x_p)$ it is clear that the image of x_p in $G_{\mathrm{ad}}(\mathbf{Q}_p)$ actually lies in $G_{\mathrm{ad}}(\mathbf{Q})$ and is independent of p. Thus $\gamma \to \gamma'$ extends to an automorphism of $G(\mathbf{R})$. Suppose there is a $\gamma \in G(\mathbf{R})$ so that

$$h(s) = y^{-1} h'(s) y \qquad s \in S(\mathbf{R}).$$

Then the map $g = (g_\infty, g_f) \to (g'_\infty y, x g_f)$ commutes with $G(\mathbf{A}_f)$ and yields a complex analytic homomorphism $\psi_{x,y}$ of each $S_K(\mathbf{C})$. I claim ψ must be of this form.

Observe by the way, that if $\delta \in G(\mathbf{Q})$ then

$$\psi_{\delta x, \delta y} = \psi_{x,y}.$$

and that if $x \in Z(\mathbf{A}_f)$ then $y \in K_\infty$ and

$$\psi_{x,y} = \psi_{x,1}.$$

Observe also that if

$$H^1\Big(\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q}), Z(\overline{\mathbf{Q}})\Big) = 1$$

or more generally, if $G(\mathbf{Q}) \to G_{\mathrm{ad}}(\mathbf{Q})$ is surjective then any x satisfying the above conditions is always of the form

$$x = \delta x'$$

with $\delta \in G(\mathbf{Q})$ and $x' \in X(\mathbf{A}_f)$.

Lemma. *Suppose ψ is a compatible family of complex-analytic homeomorphisms of the $S_K(\mathbf{C})$ which commutes with $G(\mathbf{A}_f)$. Then there is an x and a y so that*

$$\psi = \psi_{x,y}.$$

Let

$$S_K^0(\mathbf{C}) = G(\mathbf{Q}) \cap G'(\mathbf{R})K \backslash G^1(\mathbf{R})K/K_\infty K \qquad \big(G^1(\mathbf{R}) = G^0(\mathbf{R})K_\infty\big)$$

be the connected component of $S_K(\mathbf{C})$. Let s_K^0 be the point of $S_K(\mathbf{C})$ represented by 1. If ψ_K is the image of $S_K(\mathbf{C})$ defined by ψ let $\psi_K(s_K^0)$ be represented by $\big(g_\infty(K), g_f(K)\big)$. If $K \supseteq K'$ then we may take $\big(g_\infty(K'), g_f(K')\big)$ to be of the form $\big(g_\infty(K), g_f(K)k(K')\big)$ with $k(K') \in K$. As a consequence, we may take

$$\big(g_\infty(K), g_f(K)\big) = (g_\infty^0, g_f^0)$$

to be independent of K.

Moreover because of real approximation we may take $g_\infty(K) \in G^0(\mathbf{R})$.

The map ψ_K yields

$$G(\mathbf{Q}) \cap G^1(\mathbf{R})K \backslash G^1(\mathbf{R})K/K_\infty K \to G(\mathbf{Q}) \cap G^1(\mathbf{R})g_f^0 K (g_f^0)^{-1} \backslash G^1(\mathbf{R})g_f^0 K/K_\infty K.$$

Passing to covering spaces, we obtain a map

$$G^1(\mathbf{R})K/K_\infty K = G^1(\mathbf{R})g_f^0 K/K_\infty K \simeq G^1(\mathbf{R})/K_\infty,$$

which maps $1 \to (g_\infty^0, g_f^0)$. This map is independent of K. It must be of the form

$$g_\infty \to (g_\infty'' y, g_f^0)$$

where $g_\infty \to g_\infty''$ is an automorphism of $G^0(\mathbf{R})$, defined only modulo the largest normal subgroup of $G^0(\mathbf{R})$ lying in K_∞. Moreover $y \in G^0(\mathbf{R})$ and $k \to y^{-1}k''y$ takes K_∞ to itself and

$$h(s) = y^{-1}h''(s)y \qquad s \in S(\mathbf{R}).$$

Then for any $g_\infty \in G^0(\mathbf{R})$ and $g_f \in G(\mathbf{A}_p)$ the image of the point represented by (g_∞, g_f) represented by $(g_\infty' y, g_f^0 g_f)$.

For example, if $\gamma \in G(\mathbf{Q}) \cap G^0(\mathbf{R})$ and γ_f is its image in $G(\mathbf{A}_f)$ then the image of the point represented by $(\gamma, \gamma\gamma_f^{-1}) = (\gamma, 1)$ is represented by either $(y, g_f^0\gamma_f^{-1})$ or (γ_y'', g_f^0). Thus for each K there is a δ_K in $G(\mathbf{Q})$, a u_K in K_∞, and a k_K in K so that

$$\delta_K \gamma'' y u_K = y$$
$$\delta_K g_f^0 k_K = g_f^0 \gamma_f^{-1}$$

If $K \supseteq K'$

$$\delta_{K'}^{-1}\delta_K \in \gamma'' y K_\infty y^{-1}(\gamma')^{-1}$$
$$\delta_{K'}^{-1}\delta_K \in g_f^0 K (g_f^0)^{-1}$$

Thus for K sufficiently small $\delta_{K'}^{-1}\delta_K \in Z(\mathbf{Q})$. Letting K shrink we see that

$$\delta_K = g_f^0 \gamma_f^{-1} (g_f^0)^{-1}$$

in(??) $G_{\text{ad}}(\mathbf{A}_f)$. This implies that the image of g_f^0 in $G_{\text{ad}}(\mathbf{A}_f)$ actually lies in $G_{\text{ad}}(\mathbf{Q})$ and hence that if $x = g_f^0$ then

$$\gamma \to \gamma' = x\gamma x^{-1}$$

is an automorphism of $G(\mathbf{Q})$.

Set $\delta_K = \epsilon_K(\gamma')^{-1}$ with, for K small, $\epsilon_K \in Z(\mathbf{Q})$. Then

$$\epsilon_K k_K = 1$$
$$\epsilon_K(\gamma')^{-1}\gamma'' y u_K = y$$

Thus

$$\gamma'' y \in \gamma' y K_\infty.$$

By real approximation, this is true in $G^0(\mathbf{R})$; so we may suppose $\gamma'' = \gamma'$. On $G^0(\mathbf{R})G(\mathbf{A}_f)$ our map ψ is represented by

$$g = (g_\infty, g_f) \to (g'_\infty y, x g_f)$$

that is by $\psi_{x,y}$. Since

$$G(\mathbf{A}) = G(\mathbf{Q})G^0(\mathbf{R})G(\mathbf{A}_f),$$

we have

$$\psi = \psi_{x,y}$$

As a preparation for the discussion to follow, I consider the two problems for abelian groups. If $G = T$ is abelian then $S_K(\mathbf{C}) = S_K(\overline{\mathbf{Q}})$ is just a finite set of points on which $\mathfrak{G}(\overline{\mathbf{Q}}/E)$ acts. ${}^\tau S_K(\mathbf{C})$ is again a finite set of points on which $\mathfrak{G}(\overline{\mathbf{Q}}/{}^\tau E)$ acts. If $x \in S_K(\mathbf{C})$ and $\sigma \in \mathfrak{G}(\overline{\mathbf{Q}}/E)$ then

$$\tau(\sigma x) = \tau\sigma\tau^{-1}(\tau x)$$

and both τx and $\tau(\sigma x)$ lie in ${}^\tau S_K(\mathbf{C})$ the Galois groups act through their abelian quotients. The map $\sigma \to \tau\sigma\tau^{-1}$ of $\mathfrak{G}(\overline{\mathbf{Q}}/E)$ to $\mathfrak{G}(\overline{\mathbf{Q}}/{}^\tau E)$ and the map $\alpha \to \tau(\alpha)$ of I_E to $I_{{}^\tau E}$ are compatible with the isomorphisms of class field theory.

To verify that the suggested solution to the first problem is correct for T, I use the definitions of §3.9 and §3.10 of *Travaux de Shimura*. All I have to do is show that

$$\begin{array}{ccc} E^* & \xrightarrow{r'(h)} & T \\ \downarrow & & \| \\ {}^\tau E^* & \xrightarrow[r'({}^\tau h)]{} & {}^\tau T \end{array}$$

is commutative. I should perhaps be explicit about the nature of this diagram. The lattice of weights of E^* and ${}^\tau E^*$ are

$$L(E^*) = \operatorname{Ind}\Big(\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q}), \mathfrak{G}(\overline{\mathbf{Q}}/E), \mathbf{Z}\Big)$$
$$L({}^\tau E^*) = \operatorname{Ind}\Big(\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q}), \mathfrak{G}(\overline{\mathbf{Q}}/{}^\tau E), \mathbf{Z}\Big)$$

Both lattices are therefore lattices of integral-valued functions on $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$. The map $\{x_\sigma\} \to \{x'_\sigma = x_{\tau\sigma}\}$ is an isomorphism of the second module with the first. $r'(h)$ takes the weight λ of T to

$$\sigma \to \langle \sigma\lambda, \widehat{\mu}\rangle.$$

$r'({}^\tau h)$ takes λ to

$$\sigma \to \langle \sigma\lambda, \tau\widehat{\mu}\rangle.$$

Since

$$\langle \tau\sigma\lambda, \tau\widehat{\mu}\rangle = \langle \sigma\lambda, \widehat{\mu}\rangle$$

the commutativity is clear.

If E is real then the action of complex conjugation is given by an element of $E^*(\mathbf{R})$. Since $E^*(\mathbf{R}) \to T(\mathbf{R})$ and $T(\mathbf{R})$ acts trivially, the action is trivial. This is what the suggested solution to the second problem demands.

Let me now describe what seems to be the proper form of the second problem. I am not yet able to suggest a solution. Suppose we have $\varphi : (T, h) \to (G, h)$ and we have a weakly canonical model $\{S_K/E\}$ with $E \supseteq E(G, h)$. Let τ be an automorphism of $\mathbf{C}$ over E. It should be possible to show the existence of a $g \in G(\overline{\mathbf{Q}})$ and $ab \in G(\mathbf{A}_f)$ so that

(i) The homomorphism
$$\varphi' : t \to gtg^{-1}$$
from T to G is defined over $\mathbf{Q}$.

(ii) The homomorphism
$$h' : s \to g^{\tau}h(s)g^{-1}$$
from S to G over $\mathbf{R}$ is in the class of h.

(iii) The diagram
$$\begin{array}{ccc} S(T,h) & \xrightarrow{\varphi} & S(G,h) \\ \downarrow \wr & & \downarrow \tau \\ S(T, {}^{\tau}h) & & S(G,h) \\ & \searrow^{\varphi'} & \uparrow b \\ & & S(G,h) \end{array}$$
is commutative. Here b is acting through right multiplication and the isomorphism on the left is an isomorphism of sets which has been described above.

The pair (g, b) is not unique. If $\gamma \in G(\mathbf{Q})$ we could replace (g, b) by $(\gamma g, \gamma b)$. Up to such obvious changes one also wants to explicitly characterize the pair (g, b).

For the first problem, one reasonably explicit way of assigning to any automorphism τ of $\mathbf{C}$ over $\mathbf{Q}$ a cocycle $\{a_\sigma\}$ of $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ in $T(\overline{\mathbf{Q}})$, a trivialization
$$1 = b^{-1} a_\sigma \sigma(b)$$
of the cocycle in $T(\overline{\mathbf{A}}_f)$, and a d in $G(\mathbf{A}_g)$ so that

(i) If ${}^{\tau}G$ is defined as
$$\left\{ g \in G \;\middle|\; a_\sigma \sigma(g) a_\sigma^{-1} = g \right\}$$
and if
$${}^{\tau}K = \left\{ bkb^{-1} \;\middle|\; k \in K \right\}$$
then $\{{}^{\tau}S_K/{}^{\tau}E\}$ yields a canonical model for $\left\{S_{{}^{\tau}K}(\mathbf{C} : {}^{\tau}G, {}^{\tau}h)\right\}$. As before
$${}^{\tau}h(s) = h(s)$$
because $h : S \to T$ and $T(\mathbf{R}) \subseteq {}^{\tau}G(\mathbf{R})$.

(ii) Let $d' = bdb^{-1} \in {}^{\tau}G(\mathbf{A}_f)$ and let φ' be φ regarded as a map from T to ${}^{\tau}G$. It must be possible to choose the family of maps defining the canonical model
$$\psi : {}^{\tau}S_K(\mathbf{C}) \to S_{{}^{\tau}K}(\mathbf{C})$$
so that

$$\begin{array}{ccc}
S(T,h) & \longrightarrow & S(G,h) \\
\big\downarrow \wr & & \big\downarrow \tau \\
S(T,{}^\tau h) & & {}^\tau S(G,h) \\
& \searrow & \big\downarrow \psi \\
& \varphi' & S({}^\tau G,{}^\tau h) \\
& \searrow & \big\uparrow d' \\
& & S({}^\tau G,{}^\tau h)
\end{array}$$

is commutative.

Both these problems are however unsatisfying vague; some thought will have to be given to their precise formulation.

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