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Some isometric embedding and rigidity results for Riemannian manifolds

(differential system/Gauss equations)

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ABSTRACT **In this note, we announce some new results concerning rigidity and isometric embeddings of Riemannian manifolds. A special case of the main result states that a general $M^n \subset \mathbf{R}^{n+r}$, $r \le (n-1)(n-2)/2$, is uniquely determined up to finitely many constants by its induced ds^2.**

In this note, we announce some results concerning the isometric embedding and associated rigidity problem for Riemannian manifolds.

By an abstract Riemannian manifold (M, ds^2), we mean a manifold M having a Riemannian metric ds^2 given in local coordinates by $ds^2 = \Sigma_{i,j=1}^n g_{ij}(x)dx^i dx^j$. An isometric embedding

$$f: M^n \to \mathbf{R}^N \qquad [1]$$

is an embedding given locally by functions $y^a(x)$ $(a = 1,\ldots,N)$ satisfying

$$\sum_{a=1}^{N} \frac{\partial y^a(x)}{\partial x^i}\frac{\partial y^a(x)}{\partial x^j} = g_{ij}(x). \qquad [2]$$

We are interested in the two well-known questions: (*i*) How unique is a given isometric embedding? We shall refer to this as the rigidity question (cf. ref. 1 for an excellent discussion). (*ii*) Does an isometric embedding exist? This will be referred to as the existence problem. It may be posed in either the C^∞ or real analytic category.

We note that, since Eq. 2 consists of $n(n+1)/2$ equations in the N unknowns $y^a(x)$, in first approximation, we may expect an affirmative answer to the existence problem when

$$N = n(n+1)/2 \qquad [3]$$

or, equivalently, when the codimension

$$r = N - n$$

is given by

$$r = n(n-1)/2. \qquad [4]$$

Classically, the two main existence theorems are these: (*i*) The Nash embedding theorem, with refinements by several people including those given in refs. 2–4,[§] which asserts the existence of a C^∞ isometric embedding when

$$N = \begin{cases} [n(n+1)/2] + 3n + 5 & \text{global case} \\ [n(n+1)/2] + n & \text{local case.} \end{cases}$$

Unfortunately, there is no accompanying uniqueness statement. (*ii*) The Burstin–Cartan–Janet–Schläfly theorem (1, 6–8), which asserts that (*a*) a local isometric embedding exists in the embedding dimension (Eq. 3) in the real analytic case and (*b*) this embedding "depends on $(n-1)$ functions of $(n-1)$ variables," which is the expected count when one sets up Eq. 2 as a Cauchy problem. There is also a global existence assertion (5, 9). Not only does the local theorem give the hoped for statement in the real analytic case but, in the proof, there is direct interplay between the geometry of the manifold and the embedding (see the discussion at the end of ref. 6).

Turning now to the rigidity question, if we assume that the codimension $r < n(n-1)/2$ (cf. Eq. 4), then the system is "overdetermined" and one may suspect that for a general ds^2, there will be no isometric embeddings. Moreover, *if* there is an isometric embedding, then it should become "increasingly rigid" as r becomes small in comparison with n. In this regard, the main result is the classical theorem of Allendoerfer and Beez (cf. refs. 1, 8, and 10), which states that a general isometric embedding (Eq. 1) is rigid in case

$$r \leqq [n/3] \qquad [5]$$

(here, rigid means that f is determined up to Euclidean motion by the ds^2 on M).

We shall briefly explain the meaning of "general" (in ref. 10 this has a very precise meaning that we shall not give). Set $\bar{M} = f(M)$ and locally along $\bar{M}$ choose an adapted (or Darboux) frame field

$$\{x; e_1,\ldots,e_n; e_{n+1},\ldots,e_N\} = \{x; e_i; e_\mu\}$$
$$1 \leqq i,j \leqq n,\ n+1 \leqq \mu,\nu \leqq N \qquad [6]$$

for $\mathbf{R}^N$. This means that $x \in \bar{M}$, that the e_i are an orthonormal basis for the tangent space $T_x(\bar{M})$, and that the e_μ are an orthonormal basis for the normal space $N_x(\bar{M})$. If $\omega^1,\ldots,\omega^n$ is the dual coframe to the e_i, then using summation convention, we write the Riemann curvature tensor R and 2nd fundamental form H as

$$R = \frac{1}{4} R_{ijkl}\, \omega^i \wedge \omega^j \otimes \omega^k \wedge \omega^l$$

$$H = h^\mu_{ij} e_\mu \otimes \omega^i \omega^j.$$

The basic features of the embedding $\bar{M} \subset \mathbf{R}^N$ are encoded in the Gauss equations

$$\gamma(H, H) = R \qquad [7]$$

where, for $H = (h^\mu_{ij})$ and $H' = (h'^\mu_{ij})$, we define

$$\gamma(H, H')_{ijkl} = \frac{1}{2}\sum_\mu \{h^\mu_{ik}h'^\mu_{jl} + h^\mu_{jl}h'^\mu_{ik} - h^\mu_{il}h'^\mu_{jk} - h^\mu_{jk}h'^\mu_{il}\}. \qquad [8]$$

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[§] For a comprehensive survey of work on the isometric embedding problem, see ref. 5.

The main step in Allendoerfer's proof consists in an extremely ingenious computation showing that, when [5] holds and H does not satisfy a polynomial condition $P(H) = 0$, then the Gauss equations [7] uniquely determine H up to a rotation in the normal space. In this case, general means that $P(H) \neq 0$ along $\bar{M}$ (see *Appendix*, note *a*).

Our results are C^∞ and local and are therefore valid in a neighborhood of any point of M. We shall say an isometric embedding [1] *depends on constants* in case there is an integer k such that, if

$$f,f':M \rightarrow \mathbf{R}^N$$

are two isometric embeddings that agree up to (and including) order k at one point, then $f \equiv f'$ (cf. note *b*) (rigidity is essentially the case $k = 1$). We shall say that Eq. 1 is *general* in case the second fundamental form of the image does not satisfy certain polynomial relations, to be specified.

THEOREM. *Let*

$$f:M^n \rightarrow \mathbf{R}^N \qquad [9]$$

be a general isometric embedding. Then (i) *if* $r \leqq (n-1)(n-2)/2$, *the embedding depends only on constants* (note *c*); (ii) *if* $r = (n-1)(n-2)/2 + s$, $0 \leqq s \leqq n-1$, *then* [9] *depends on, at most, functions of* s *variables; and* (iii) *if*

$$\begin{cases} r \leqq n & n \geqq 6 \\ r \leqq 3 & n = 4 \\ r \leqq 4 & n = 5, \end{cases}$$

then the embedding 9 *is rigid* (note *d*).

To better understand the codimension ranges in the theorem, we remark that the embedding codimension (Eq. 4) is given by

$$n(n-1)/2 = (n-1)(n-2)/2 + (n-1).$$

In addition, we have an existence theorem. To explain it, we first remark that our proof of the above theorem will show that, when $r < n(n-1)/2$, Eq. 2 is in a precise mathematical sense overdetermined. In so doing, we will show that, if an embedding [1] exists, then for each point of M, the sequence

$$R, \nabla R, \nabla^2 R, \nabla^3 R, \ldots, \qquad [10]$$

consisting of the curvature and its successive covariant derivatives, satisfies polynomial relationships

$$Q_\lambda(R, \nabla R, \ldots, \nabla^k R) \equiv 0, \quad \lambda = 1,2, \ldots, d(r) \qquad [11]$$

[$k = k(r)$]. We denote by V_r the variety given by [11]. Although this variety is complicated, we are able to determine it explicitly when the codimension $r = 1,2$ (note *e*) and to determine its tangent space in general.

PROPOSITION 1. *Suppose that* (M, ds^2) *is an abstract Riemannian manifold whose curvature sequence* [10] *lies in* $V_r - V_{r-1}$ *for each* $x \in M$. *Then if*

$$r \leqq (n-1)(n-2)/2, \qquad [12]$$

there is a local C^∞ *isometric embedding*.

This result is only a proposition, because the main part of the proof already appears in the theorem and consists in reducing the partial differential equation system (Eq. 2) to a succesion of ordinary differential equations.

This proposition was previously obtained for low codimension by Thomas (11) for the case $r = 1$ and Allendoerfer (10) for the case $r \leq [n/4]$. A more modern discussion of their results has been given by Chern and Osserman (12).

We would like to briefly remark on the proof of our theorem.

As a general setting, we use Cartan's theory of exterior differential systems (refs. 6, 13, 14 and unpublished data). This theory gives a precise meaning to the sense in which [1] is overdetermined for $r < n(n-1)/2$ and, at least in principle, gives a method for determining the "integrability conditions" [11] that must be satisfied by a ds^2 in order that a formal germ of isometric embedding [2] exist around each point of M. We emphasize that having a formalism that encompasses both the geometry in the problem as well as exhibiting in computable form the algebra underlying the integrability conditions is essential to the problem (note *f*).

Given this general setting, our argument first puts the differential system underlying the isometric embedding system [2], together with its prolongations, in a good form suitable to the particular problem (the procedure is somewhat different from that used in the existing proofs of the Burstin–Cartan–Janet–Schläfly theorem). Then we combine the classical methods of Cartan (6) with the modern formulations from refs. 13 and 14 in a form adapted to the present problem (note *g*). To explain this, we set

$$\begin{cases} W = N_x(\bar{M}) \cong \mathbf{R}_r \\ V = T_x(\bar{M}) \cong \mathbf{R}^n \\ K = \{\text{space of curvature-like tensors}\} \subset \otimes^4 V^*. \end{cases}$$

By our first step, we are led to consider the linearized and prolonged version

$$\gamma_H{}^{(q)}: W \otimes \mathrm{Sym}^{q+2} V^* \rightarrow K \otimes \mathrm{Sym}^q V^* \qquad [13]$$

of the Gauss Eq. 7, where for

$$H = h^\mu_{ij} e_\mu \otimes \omega^i \omega^j \in W \otimes \mathrm{Sym}^2 V^*$$

$$H' = h'^\mu_{i_1 \ldots i_{q+2}} e_\mu \otimes \omega^{i_1} \ldots \omega^{i_{q+2}} \in W \otimes \mathrm{Sym}^{q+2} V^*,$$

the mapping 13 is given by

$$\gamma_H{}^{(q)}(H')_{ijkl} = \frac{1}{2} \sum_\mu \{h^\mu_{ik} h'^\mu_{jlm_1 \cdots m_q} + h^\mu_{jl} h'^\mu_{ikm_1 \cdots m_q} - h^\mu_{il} h'^\mu_{jkm_1 \cdots m_q} - h^\mu_{jk} h'^\mu_{ilm_1 \cdots m_q}\} \omega^{m_1} \ldots \omega^{m_q}. \qquad [14]$$

We note that

$$\frac{\partial}{\partial \omega^i} (\gamma_H{}^{(q)}(H')) = \binom{q+2}{q} \gamma_H{}^{(q-1)} \left(\frac{\partial H'}{\partial \omega^i} \right). \qquad [15]$$

This suggests that we set

$$\mathrm{Sym}^{\bullet} V = \bigoplus_{q \geqq 0} \mathrm{Sym}^q V$$

$$\mathrm{Sym}^{\bullet+2} V = \bigoplus_{q \geqq 0} \mathrm{Sym}^{q+2} V$$

and consider the duals $\gamma_H{}^{(q)*}$ of the maps [13] collectively as giving

$$\gamma_H^{\bullet *}: K^* \otimes \mathrm{Sym}^{\bullet} V \rightarrow W^* \otimes \mathrm{Sym}^{\bullet+2} V \qquad [16]$$

where $\gamma_H^{\bullet *} = \bigoplus_{q \geqq 0} \gamma_H{}^{(q)*}$. From Eq. 15, it is easy to conclude that, aside from irrelevant scaling factors, $\gamma_H^{\bullet *}$ is a map of graded $\mathrm{Sym}^{\bullet} V$-modules.

Now it is well known that, when viewed globally, the Gauss Eq. 7 and their prolongations [13] present severe algebraic difficulties (cf. refs. 8 and 15). For instance, suppose we consider the case $n = 4$, $r = 2$ of an

$$\bar{M}^4 \subset \mathbf{R}^6.$$

Then

$$\dim W\otimes \mathrm{Sym}^2V^* = 20$$

$$\dim K = 20$$

[in general, $\dim K = n^2(n^2 - 1)/12$], and the Gauss map [7] is a quadratic map

$$\gamma{:}\mathbf{R}^{20} \to \mathbf{R}^{20}.$$

Since we can rotate in the normal space,

$$\dim \gamma^{-1}(R) \geqq 1$$

for all $R \in$ image γ. By using Young tableaux to study the associated bilinear map [8] we are able to prove the

PROPOSITION 2. *For any* $\mathrm{R} \in$ *image* γ,

$$\dim \gamma^{-1}(\mathrm{R}) \geqq 2. \qquad [17]$$

Thus the Gauss map has "hidden symmetries."

As another example to illustrate that a simple counting of constants in the Gauss equations and their prolongations is misleading, we denote by

$$K^{(q)} \subset K\otimes \mathrm{Sym}^qV^*,$$

the space of tensors with the symmetries of the principal part of a general ∇^qR (cf. ref. 8 for this notation). Then, it is immediate that

$$\gamma_H{}^{(q)}{:}\mathrm{W}\otimes \mathrm{Sym}^{q+2}\mathrm{V}^* \to K^{(q)}. \qquad [18]$$

Now

$$\dim K^{(q)} = \frac{n(n-1)}{2}\,\frac{q+1}{q+3}\binom{N+q+1}{q+2} \approx \frac{n(n-1)}{2}\,\frac{q^{n-1}}{(n-1)!}$$

$$\dim \mathrm{W}\otimes \mathrm{Sym}^{q+2}\mathrm{V}^* = r\binom{n+q+1}{q+2} \approx r\,\frac{q^{n-1}}{(n-1)!},$$

so that, if $r < n(n-1)/2$, the mappings [18] cannot be surjective for large q (note *h*). However, even when the conditions

$$r < n(n-1)/2,\ q \geqq 1, \text{ and } \dim \mathrm{W}\otimes \mathrm{Sym}^{q+2}\mathrm{V}^* \leqq \dim \mathrm{K}^{(q)}$$

are satisfied, so that the mapping has no "obvious" kernel, it may still happen that $\gamma_H{}^{(q)}$ fails to be injective. For example, we have the

PROPOSITION 2. *For a general* $\mathrm{M}^3 \subset \mathbf{R}^5$ and for all $q \geqq 1$,

$$\dim\,[\ker \gamma_H{}^{(q)}] = 6. \qquad [19]$$

In summary, it would seem that treating the Gauss equations and their prolongations by direct, global (in the cotangent spaces) methods presents a formidable task (cf. the complicated computations in ref. 8).

Now it is well known (cf. ref. 16) that *over* $\mathbf{C}$ the data [16] are essentially equivalent to giving coherent sheaves $\mathcal{K}^*$, $\mathcal{W}^*(2)$ over $\mathbf{P}V^*_{\mathbf{C}} \cong \mathbf{P}^{n-1}_{\mathbf{C}}$ together with a sheaf map

$$\Gamma_H{:}\mathcal{K}^* \to \mathcal{W}^*(2). \qquad [20]$$

To explain this dictionary in one direction, for large q, we have

$$H^0(\mathbf{P}^{n-1}_{\mathbf{C},}\mathcal{K}^*(q)) \cong K^* \otimes \mathrm{Sym}^qV^*\otimes\mathbf{C}$$

$$H^0(\mathbf{P}^{n-1}_{\mathbf{C},}\mathcal{W}^*(q+2)) \cong W^* \otimes \mathrm{Sym}^{q+2}V^*\otimes\mathbf{C}$$

and the induced map on global sections is $\gamma_H{}^{(q)*}$. This suggests that we localize Eq. 8 at vectors $0 \neq \xi \in V^*_{\mathbf{C}} = V^* \otimes \mathbf{C}$. When this is done, a minor miracle occurs, in that the *localized* Gauss equations become extremely simple. For example, when $\xi = (0,0, \ldots, 1)$, the localization of [8] is essentially given by

$$(h'^{\mu}_{ij})_{1\leqq i,j\leqq n-1} \to \sum_{\mu} h^{\mu}_{nn}\, h'^{\mu}_{ij}. \qquad [21]$$

A straightforward dimension count shows that

$$\mathrm{codim}\begin{Bmatrix} H \in W \otimes \mathrm{Sym}^2V^*{:}(21) \\ \text{fails to be injective} \\ \text{for some } 0 \neq \xi\in V_{\mathbf{C}} \end{Bmatrix} = (n-1)(n-2)/2 - r + 1. \qquad [22]$$

Together with more or less standard reasoning from algebraic geometry, [22] leads to

PROPOSITION 4. *If* $\mathrm{r} \leqq (\mathrm{n}-1)(\mathrm{n}-2)/2$ *and* $\mathrm{H} \in \mathrm{W}\otimes \mathrm{Sym}^2\mathrm{V}^*$ *is general, then* (i) *sheaf mapping* [20] *is surjective and* (ii) *for* $\mathrm{q} \geqq \mathrm{q}_0$, *the mappings* $\gamma_{\mathrm{H}}{}^{(q)}$ *in* [14] *are injective* (note *i*). Intuitively, (*ii*) means the "Taylor's series" or jet of the 2nd fundamental form is uniquely determined, up to finitely many terms, by sequence [10], and this underlies part *i* in the *Theorem.*

Part *ii* is similar, but part *iii* requires a deeper argument. For this what we prove is the

PROPOSITION 5. *Under the conditions in part* iii *of theorem, sequence* [10] *uniquely determines the 2nd fundamental form at each point of* M.

As we saw when $n = 4$, $r = 2$ (cf. proposition 2), this is false for just the curvature (i.e., the Gauss Eq. 7 may not be uniquely solvable, up to normal rotations).

Even (and perhaps especially) when $r = n(n-1)/2$ is the embedding codimension, our method has interest. From [20], we have

$$\mathcal{K}^* \xrightarrow{\Gamma_H} \mathcal{W}^*(2) \to \mathcal{F} \to 0, \qquad [23]$$

where the cokernel $\mathcal{F}$ is a coherent sheaf on $\mathbf{P}^{n-1}_{\mathbf{C}} = \mathbf{P}[\mathrm{T}^*_x(M) \otimes \mathbf{C}]$. As a generalization of *Proposition 4*, we prove that, if H is general,

$$\dim(\mathrm{supp}\,\mathcal{F}) = \max\{-1, r - [(n-1)(n-2)/2] - 1\}$$

(the number ultimately comes from [22]. Moreover, we may identify supp $\mathcal{F}$ with the scheme of complex Monge characteristics for the isometric embedding system (note *j*). Over $\mathbf{R}$, these characteristics turn out to be the *asymptotic codirections* for a given solution to the Gauss equations.

In particular, we consider the case $r = n(n-1)/2$. The isometric embedding system is involutive, and the space X of integral elements may be thought of as the variety of solutions to the Gauss Eqs. 7. Over each point $H \in X$, the characteristic variety (we drop the word scheme)

$$\Xi_H \subset \mathbf{P}V^*\ [\cong \mathbf{P}\,T^*_x(M)]$$

is the support of the sheaf $\mathcal{F}$. The integral element H is ordinary (so that the Cartan–Kähler theorem applies in the real analytic case) exactly when

$$\Xi_H \neq \mathbf{P}V^*.$$

We shall denote by $\Xi_{H,\mathbf{C}} \subset \mathbf{P}V^*_{\mathbf{C}}$ the complexified characteristic variety.

PROPOSITION 6. (i) *If* H *is ordinary, then*

$$deg\ \Xi_{\mathrm{H,C}} = \mathrm{n}$$

(ii) *If* $\mathrm{n} \geq 3$, *then* Ξ_{H} *is a nonempty real hypersurface* (*of degree* n).

Thus, the isometric embedding system is hyperbolic except for the classical case $n = 2$ when the Gaussian curvature is positive (note *k*). Moreover, the characteristic variety has a canonical (real) resolution of singularities (these occur in codimension 4). This picture may be expected to have great bearing on the propagation of singularities behavior of the isometric embedding system.

Example: Suppose we have a general $M^3 \subset \mathbf{R}^6$. Then at each point $x \in M$, the characteristic variety of the isometric embedding is a smooth real cubic curve $C_x \subset \mathbf{P}T^*_x(M)$. The Monge cones are just the dual algebraic curves $C^*_x \subset \mathbf{P}T_x(M)$ (these have degree 6 with 9 cusp singularities), and the bicharacteristic curves project to curves $x(t)$ in M whose tangent direction $x'(t) \in C^*_{x(t)}$.

As a final illustration of how algebro-geometric considerations enter, we consider over $\mathbf{P}V^*$ the vector bundle E whose fiber over $0 \neq \omega \in V^*$ is given by

$$E_\omega = \mathrm{Sym}^2 \omega^\perp .$$

It can be shown that [23] is (essentially)

$$0 \to \mathrm{E} \xrightarrow{\Gamma_H} W^*(2) \to \mathcal{F} \to 0.$$

From this, it follows that the 2nd fundamental form $H \in W \otimes \mathrm{Sym}^2 V^*$ is uniquely determined, up to $GL(W)$, by the characteristic variety sheaf. Under additional general position assumptions, H is determined by Ξ up to rotations in W. When $n = 3$, this refines the theorem of Tenenblat (17) (actually, the result of ref. 17 is not quite correct as it stands).

APPENDIX

(*a*) The example of a cylinder ($M^n = M^k \times \mathbf{R}^{n-k}$, where $M^k \subset \mathbf{R}^{N-n+k}$) shows that rigidity may fail with no assumption on generality.

(*b*) Briefly, f is uniquely determined by its *k-jet* at one point of M.

(*c*) Intuitively, viewing M as a sheet of metal, it may be bent, but only in a finite parameter way. In particular, the moduli space for local isometric embeddings is in this case finite dimensional. This is the first example we have seen of such a phenomenon occuring naturally in a geometric problem.

(*d*) The meaning of "general" is somewhat more subtle here than in *i* and *ii* and will not be explained here.

(*e*) The case $r = 1$ is classical (cf. refs. 18 and 19). A consequence of our result when $r = 2$ is that $k(2) = 1$; i.e., the conditions under which an (M^n, ds^2) can be isometrically embedded in $\mathbf{R}^{n+2}$ are given by explicit polynomial relations on R and ∇R [in this regard, we note that $k(1) = 0$].

(*f*) In this regard, we would like to emphasize the influence, both personal and mathematical, of the formalism for studying overdetermined systems introduced by Spencer (13). In particular, refs. 7, 8, and 20 study the isometric embedding problem by using prolongations and involution formulated in terms of Spencer cohomology [cf. also Goldschmidt (21)]. In refs. 14 and 22, Spencer's homological formalism was dualized and then the standard machinery of commutative algebra becomes directly applicable (cf. also refs. 23 and 24). Although we have chosen to use moving frames, the Goldschmidt–Spencer computational formalism could have been equally well applied (in the setting of ref. 7).

(*g*) Roughly speaking, this technique may be applicable whenever a geometric configuration is automatically the solution to an overdetermined partial differential equation system [such as an $M^n \subset \mathbf{R}^N$, $N < n(n + 1)/2$].

(*h*) This observation may be used to show that [2] fails to be involutive below the embedding dimension.

(*i*) The passage from *i* to *ii* is a sort of "module Nullstellensatz."

(*j*) However, the structure sheaf of the characteristic scheme only coincides with $\mathcal{F}$ up to codimension 4. We strongly suspect that the facts that supp $\mathcal{F}$ is Cohen–Macaulay and has an invertible dualizing sheaf will be analytically significant. We use the word Monge characteristic to distinguish from Cauchy characteristics and Cartan characteristics (= singular integral elements), both of which are used in ref. 6.

(*k*) In particular, it is not elliptic except for the case of a surface $f: M^2 \to \mathbf{R}^3$ whose Gaussian curvature is positive—i.e., $R > 0$ (cf. ref. 1). This phenomenon was known to S. S. Chern and H. Levy.

Note Added in Proof. The relationships discussed in note *e* have been obtained by G. Kallo.

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