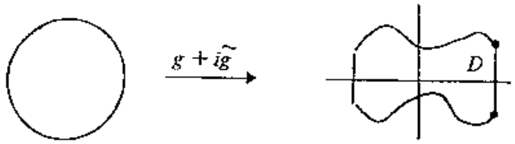

and to make use of the fact that $g+i\tilde{g}$ maps the disk *univalently* onto a domain D with strong vertical symmetry. A theorem of Lehto's asserts that N is subharmonic in $C-\{f(0)\}$, and so, by a variant of Theorem A, N^* is subharmonic in C^+, except for a certain correction term. The symmetry of D causes $\bar{N}^*$ to be harmonic in D^+, except for the same correction term. Now one uses the maximum principle, together with some other facts, to prove the majorization $N^* \leqq \bar{N}^*$, which implies (6).

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Some Problems in Complex Analytic Geometry with Growth Conditions

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The purpose of this talk is to raise a few questions in the general area of complex-analytic geometry with growth-conditions.

1. Notations. We shall be concerned with a complex manifold M of the form

$$M=\bar{M}-N$$

where $\bar{M}$ is an n-dimensional compact complex manifold and N is a k-dimensional complex submanifold. We assume given a metric on $\bar{M}$ and an exhaustion function

$$\tau: M\to \mathbf{R}$$

such that near N we have approximately

$$\tau(p)\sim -\log\delta(p,N)$$

where $\delta(p,N)$ is the distance from M to N. If we set $M[r]=\{pM:\tau(p)\leqq \log r\}$ then the Levi form

$$L(\tau)=\sqrt{-1}\,\partial\bar{\partial}\tau/2$$

in the holomorphic tangent spaces to $\partial M[r]$ will, for large r, have $\geqq n-k-1$ negative eigenvalues in the directions normal to N. The sign of the remaining eigenvalues will depend on the curvature in the normal bundle to N.

A prototypical example is when

$$\bar{M}=\mathbf{P}^n,\quad N=\mathbf{P}^k\quad\text{and}\quad M=\mathbf{P}^n-\mathbf{P}^k.$$

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Setting $n=m+k+1$ we may consider P^n as the join

(1)
$$P^n = P^m + P^k$$

corresponding to the direct sum decomposition $C^{n+1}=C^{m+1}\oplus C^{k+1}$, and use homogeneous coordinates

$$[z, w] = [z_0, \ldots, z_m;\ w_0, \ldots, w_k]$$

on P^n. Then we may take

$$\tau([z, w]) = \log \|w/z\|$$

for an exhaustion function. For $k=n-1$, M is C^n with Euclidean coordinates $(\zeta_1, \ldots, \zeta_n)=(w_0/z_0, \ldots, w_{n-1}/z_0)$ and $\tau(\zeta)=\log\|\zeta\|$.

We will be interested in the asymptotic growth properties of analytic and meromorphic functions, holomorphic vector bundles and their sections, analytic subvarieties, etc. as we go to infinity in M, cf. [4] and [12]. Thus we are studying the behavior of essential singularities of analytic objects along an analytic subvariety, as opposed to the rather different and more difficult questions of singularities along the real $(2n-1)$-dimensional boundary of a domain.

2. The Bezout problem. The growth of an analytic subvariety $V\subset M$ will be measured by

$$\mu(V, r) = \mathrm{vol}\,(V[r])$$

where $V[r]=V\cap M[r]$ and $\mathrm{vol}\,(V[r])$ is the volume of $V[r]$ relative to the given metric on $\overline{M}$. We recall that

$$\mathrm{vol}\,(V[r]) = \frac{1}{k!}\int_{V[r]} \varphi^k$$

where φ is the $(1,1)$ form associated to the metric (Wirtinger theorem). It is a basic theorem due to Bishop and Stoll (cf. [10]) that V has a removable singularity along N; i.e., $\overline{V}$ is an analytic subvariety of $\overline{M}$, if and only if $\mu(V, r)$ is bounded. The transcendental Bezout problem is to estimate the growth of the intersection $V\cap W$ in terms of the growth of the analytic subvarieties V and W of M.

The problem arises already when M is C^n. By the diagonal construction we may reduce to the case where W is a linear space (cf. [6]). Then the Bezout estimate holds in case V is a hypersurface, but fails when codim $V\geqq 2$. Thus, for an analytic curve in C^2 the growth of the number of points of intersection with a line is estimated by the growth of the area of the curve, but Cornalba and Shiffman [5] gave an analytic curve V in C^3 where the corresponding statement is false. If we let

$$\Gamma = \overline{V}\cap P^2$$

be the limits of the asymptotic directions $\overline{0p}$ as $p\in V$ tends to infinity, then the intuitive reason for the failure of Bezout seems to be the somewhat arbitrary character of Γ; in any case, it certainly need not be evenly distributed. The Bezout estimate is concerned with the intersection properties of a neighborhood of Γ with lines in

the P^2 at infinity in C^3, and there have been estimates on this intersection in terms of $\mu(V, r)$ for almost all lines (cf. Carlson [3] and Gruman [8]), and in terms of $\mu(V, r)$ together with the growth of the osculating spaces associated to V (cf. [6] and Stoll [11]).

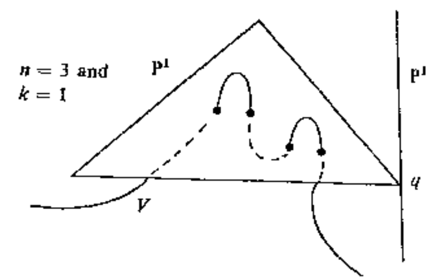
In his Harvard thesis [2], Moshe Breiner has clarified the Bezout problem and to some extent shed light on the general character of essential singularities of analytic varieties. To explain what he did we take

$$M = P^n - P^l$$

and recall that for a k-dimensional analytic subvariety $V\subset M$ the Remmert–Stein theorem [9] implies that V is algebraic if either

$$k > l, \quad \text{or} \quad k = l \quad \text{and} \quad \overline{V}\cap P^k \quad \text{omits an open set.}$$

We take the case $k=l$ and set $n=m+k+1$ so that the decomposition (1) holds.



For each point $q\in P^k$ the linear span $\overline{P^m; q}$ of P^m and q is a $P^{n-k}(q)$ and a special case of the main theorem in [2] is: *Given $\alpha>1$ there exists $C_\alpha>0$ such that for any $V_k\subset P^n-P^k$ and any $q\in P^k$*

(2)
$$\mu(V\cap P^{n-k}(q), r) \leqq C_\alpha \mu(V, \alpha r).$$

For $k=1$ we recover the aforementioned Bezout theorem for analytic hypersurfaces in C^n. We may informally paraphrase his result by saying that the analytic Bezout theorem holds in the first dimension in which the Remmert–Stein theorem allows an essential singularity.

Breiner's proof uses integral formulas from Nevanlinna theory and we would like to discuss intuitively what he does. For each point q we define the attraction of V to q to be measured by the area of $V[r]\cap U$ where U is a neighborhood of q in $\overline{M}$. Then Breiner's argument gives that the attraction to any particular q is bounded by the average attraction to all points of P^k, and by integral geometry this average attraction turns out to essentially be the area of $V[r]$. In particular it follows that the attraction of V to q is the same for almost all points of P^k, and so the erratic behavior encountered in the set of asymptotic directions to a curve in C^3 is precluded (this erratic behavior is even more evident for points in C^2).

Now his result about the attraction of V to points $q\in N$ is undoubtedly true for general situations $V\subset M=\bar{M}-N$ provided that $\dim V\geqq \dim N$. Still likely but somewhat less evident is the

Question. Is the attraction of V_k to $q\in N$ equidistributed provided that for large r the Levi form $L(\tau)$ has $\geqq n-k-1$ negative eigenvalues in the holomorphic tangent spaces to $\partial M[r]$?

For example, if we let M be the usual quadratic transformation of $\boldsymbol{P}^n$ along $\boldsymbol{P}^k$ and N the total transform of $\boldsymbol{P}^k$, then N is the projectivized normal bundle to $\boldsymbol{P}^k$ in $\boldsymbol{P}^n$, and the question asks not only about the attraction of V to points of $\boldsymbol{P}^k$ but also about the normal component of the tangent spaces to V.

Another question we should like to discuss briefly arises from the use of curvature integrals to measure growth. For an entire analytic set $V_k\subset \boldsymbol{C}^n$ we consider the Gauss map

$$\gamma\colon V\to G(k,n)$$

that assigns to each smooth point $p\in V$ the complex tangent plane $T_p(V)$ viewed as lying in the Grassmannian $G(k,n)$ of k-planes through the origin in $\boldsymbol{C}^n$. Clearly γ extends to a meromorphic mapping on all of V, and at the smooth points the usual curvature matrix Ω_V of V is the negative transpose of the pullback under γ of the curvature in the universal subbundle over the Grassmannian. The Chern forms $c_k(\Omega_V)$ are defined by

$$\det\left(tI+\frac{\sqrt{-1}}{2\pi}\,\Omega_V\right)=\sum_{l=0}^{k}(-1)^l t^{k-l}c_l(\Omega_V).$$

We denote by φ the standard Kähler form on $\boldsymbol{C}^n$, and recall that $(1/k!)\int_{V[r]}\varphi^k$ is the Euclidean area $\operatorname{vol}(V[r])$ of $V[r]$ and

$$\mu_0(V,r)=\frac{1}{k!\,r^{2k}}\int_{V[r]}\varphi^k=\frac{\operatorname{vol}(V[r])}{r^{2k}}$$

is an increasing function of r with

$$\lim_{r\to 0}\mu_0(V,r)=\operatorname{mult}_0(V)$$

being the multiplicity of V at the origin. In [7] it is proved that the expressions

$$\mu_l(V,r)=\frac{1}{(k-l)!\,r^{2(k-l)}}\int_{V[r]}c_l(\Omega_V)\wedge\varphi^{k-l}$$

are well defined and increasing in r, and whose limits as $r\to 0$ have to do with the singularity structure of V at the origin. On the other hand, the quantities

$$r^{2(k-l)}\mu_l(V,r)=\frac{1}{(k-l)!}\int_{V[r]}c_k(\Omega_V)\wedge\varphi^{k-l}$$

are the coefficients in the expansion in powers of ε of the volume of the ε-tube

$\tau_\varepsilon(V,r)=\{q\in\boldsymbol{C}^n\colon \delta(q,V[r])\leqq\varepsilon\}$ around $V[r]$. These integrals may be thought of as measuring the growth of the currents obtained by the standard smoothing of the current defined by integration over V, and as such may be expected to play a role in such questions as extending functions from V to $\boldsymbol{C}^n$ preserving growth conditions. Here we should like to pose the

Question. Is there a Bezout estimate for the refined growth indicator

$$\mu(V,r)=\sum_{l\geqq 0}\mu_l(V,r)?$$

We remark that the analogue of Crofton's formula

$$\mu_0(V,r)=\int_{A\in G(n-k+l,n)}\mu_0(A\cap V,r)\,dA$$

ts provided by the kinematic formula given in [7], so that an affirmative answer to ihis question would follow from plurisubharmonic properties of the elementary symmetric functions of the 2nd fundamental form of V in $\boldsymbol{C}^n$.

3. Representing homology classes by analytic cycles. Recall that an analytic cycle Z on a complex manifold is a locally finite formal sum $\sum_i n_i Z_i$ of irreducible analytic varieties with integer coefficients. The growth of Z will mean that of the analytic variety $|Z|=\sum_i |n_i| Z_i$. If Z has pure dimension $n-k$ there is the fundamental class ($\boldsymbol{Q}$-coefficients)

$$\eta_Z\in H_{2n-2k}(M)\cong H^{2k}(M).$$

A long-standing general problem is how much of $H^{2k}(M)$ is represented by such fundamental classes? When M is a compact algebraic variety there is the famous Hodge conjecture. At the opposite extreme, when M is Stein a theorem of Grauert implies that all of $H^{2k}(M)$ is represented by analytic cycles. Here the natural analogue of the Hodge conjecture is to impose growth conditions on the cycles.

In general we may look for restrictions on η_Z imposed by Hodge theory. Suppose we denote by $H^{2k}_{\mathrm{DR}}(M)$ the complex deRham cohomology and recall the Hodge filtration $F^pH^{2k}_{\mathrm{DR}}(M)$ that may be defined as follows:

We consider the usual double complex

$$A^*(M)=\bigoplus_{p,q}A^{p,q}(M)$$

obtained by decomposing the C^∞ forms into (p,q) type and writing $d=\partial+\bar\partial$. The associated total complex is the deRham complex, and the Fröhlicher spectral sequence has

$$E_1^{p,q}\cong H^q_{\bar\partial}(M,\Omega^p),\quad E_\infty\Rightarrow H^*_{\mathrm{DR}}(M).$$

The Hodge filtration is that induced on the abutment of the E_∞ term. When M is a compact Kähler manifold, $E_1=E_\infty$ and the Hodge filtration is

$$F^pH^{2k}(M)=H^{2k,0}(M)\oplus\ldots\oplus H^{p,2k-p}(M)$$

where $H^l_{\mathrm{DR}}(M)=\oplus_{r+s=l}H^{r,s}(M)$ is the Hodge decomposition on cohomology. When M is Stein $E_1^{p,q}=0$ for $q>0$ and all $F^pH^{2k}_{\mathrm{DR}}(M)=H^{2k}_{\mathrm{DR}}(M)$.

Alternatively, we consider the holomorphic deRham complex

$$0\to \mathbf{C}\to\Omega^0\xrightarrow{d}\Omega^1\xrightarrow{d}\dots\xrightarrow{d}\Omega^n\to 0.$$

By the holomorphic Poincaré lemma this complex separates into short exact sequences

$$0\to\mathbf{C}\to\Omega^0\xrightarrow{d}\Omega^1_c\to 0,$$
$$0\to\Omega^1_c\to\Omega^1\xrightarrow{d}\Omega^2_c\to 0,$$
$$\vdots$$
$$0\to\Omega^{n-1}_c\to\Omega^{n-1}\xrightarrow{d}\Omega^n\to 0,$$

where Ω^q_c is the sheaf of closed holomorphic q-forms. In cohomology we have

$$H^{2k-1}(M,\Omega^1_c)\to H^{2k}(M,\mathbf{C})\to H^{2k}(M,\Omega^0)$$
$$H^{2k-2}(M,\Omega^2_c)\to H^{2k-1}(M,\Omega^1_c)\to H^{2k-1}(M,\Omega^1)$$
$$\vdots$$
$$H^0(M,\Omega^{2k}_c)\to H^1(M,\Omega^{2k-1}_c)\to H^1(M,\Omega^{2k-1}) \tag{3}$$

A class $\eta\in H^{2k}_{\mathrm{DR}}(M)$ is in $F^pH^{2k}_{\mathrm{DR}}(M)$ if it is in the image of $H^{2k-p}(M,\Omega^p_c)$ in (3). A basic fact is that $\eta_Z\in F^kH^{2k}_{\mathrm{DR}}(M)$ for any analytic cycle Z. When M is Stein this imposes no conditions and Grauert's theorem provides the existence theorem.

Suppose now that $\overline{M}\subset\mathbf{P}^N$ is a projective algebraic manifold, $N=\overline{M}\cdot\mathbf{P}^{N-k}$ is a linear section of $\overline{M}$, and $M=\overline{M}-N$. For example, when $k=1$, M is an affine algebraic variety and hence a Stein manifold. If $\eta\in H^{2k}_{\mathrm{DR}}(\overline{M})$ restricts to $\eta\in H^{2k}_{\mathrm{DR}}(M)$ then we have seen that $\eta=\eta_Z$ for a generally transcendental analytic cycle Z in M. In fact, it is possible to provide a lower bound on the transcendence level of Z as follows [4]: If $\bar\eta$ is primitive and $\bar\eta^{k+l,k-l}\neq 0$ in the Hodge decomposition, then

$$\mu(|Z|,r)\gg C^{re}r^l \tag{4}$$

where C^{re} is a positive constant. In other words $|Z|$ must be of finite order $\geq l$. It was also proved in [4] that the estimate (4) is sharp in case $k=1$. Intuitively the reason that we were able to establish this had to do with the fact that the analytic Bezout theorem is valid in the codimension one case, and consequently the analytic formalism goes well.

Now, recalling that for general k,

$$M=\overline{M}-\overline{M}\cdot\mathbf{P}^{N-k}$$

a theorem of Andreotti–Grauert [1] gives

$$H^p(M,\Omega^{2k-p})=0\quad\text{for}\quad p>k,$$

so that

$$H^k(M,\Omega^k_c)\to H^{2k}_{\mathrm{DR}}(M)$$

is surjective; i.e., $F^kH^{2k}_{\mathrm{DR}}(M)=H^{2k}_{\mathrm{DR}}(M)$ in this case. Consequently there are no Hodge-theoretic objections to representing all of $H^{2k}(M)$ by analytic cycles in this particular dimension; note that this is exactly the dimension where the Remmert–Stein theorem first allows transcendental analytic varieties. We we may ask the

Question 3. For $M=\overline{M}-\overline{M}\cdot\mathbf{P}^{N-k}$ as above, is all of $H^{2k}(M)$ represented by analytic cycles? Can we choose these cycles to have finite order $\leq k$?

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