projective spaces. A. Andreotti and Frankel have already established this in dimension 4.[9] In dimension 6, it is not yet known whether a compact Kaehler manifold of positive curvature is homologically complex projective space.

Complete proofs and details will be presented elsewhere.

* The research of this author was supported by the Air Force Office of Scientific Research.

[1] Berger, M., "Pincement Riemannien et pincement holomorphe," *Ann. Scuola norm. super. Pisa III*, **14**, 151–159 (1960); see also "Correction d'une article antérieur," *ibid.*, **16**, 297 (1962).

[2] Berger, M., "On the characteristic of positively-pinched Riemannian manifolds," these PROCEEDINGS, **48**, 1915–1917 (1962).

[3] Bochner, S., "Euler-Poincaré characteristic for locally homogeneous and complex spaces," *Ann. Math.*, **51**, 241–261 (1950).

[4] Chern, S.-S., "On curvature and characteristic classes of a Riemannian manifold," *Hamb. Math. Abh.*, **20–21**, 117–126 (1955).

[5] Goldberg, S. I., *Curvature and Homology* (New York: Academic Press, 1962).

[6] Guggenheimer, H., "Vierdimensionale Einsteinräume," *Rend. di Mat.*, **11**, 1–12 (1952).

[7] Samelson, H., "On curvature and characteristic of homogeneous spaces," *Mich. Math. J.*, **5**, 13–18 (1958).

[8] Tsukamoto, Y., "On Kählerian manifolds with positive holomorphic sectional curvature," *Proc. Japan Acad.*, **33**, 333–335 (1957).

[9] Frankel, T., "Manifolds with positive curvature," *Pac. J. Math.*, **11**, 165–174 (1961).

[10] Theorem 2, with the exception of the Corollary, was announced by Berger in an invited address at the International Congress of Mathematicians, Stockholm, 1962.

SOME REMARKS ON AUTOMORPHISMS, ANALYTIC BUNDLES, AND EMBEDDINGS OF COMPLEX ALGEBRAIC VARIETIES

BY PHILLIP A. GRIFFITHS

84 GYPSY LANE, BERKELEY, CALIFORNIA

Communicated by D. C. Spencer, April 2, 1963

1. Let X be a compact, connected complex manifold (nonsingular), let $A(X)$ be the complex Lie group of analytic automorphisms of X, and $A^{\circ}(X) = A$ the identity component of $A(X)$. Furthermore, let $E \rightarrow \mathbf{E} \rightarrow X$ be an analytic vector bundle arising from an analytic principal bundle $G \rightarrow P \rightarrow X$ by a linear action of the complex Lie group G on a complex vector space E. Let $\mathcal{E}$ be the sheaf of germs of holomorphic cross sections of $\mathbf{E}$; denote by Ω the sheaf associated to the trivial bundle, and by Θ the sheaf associated to the holomorphic tangent bundle $\mathbf{T}_X$ of X. Associated to $G \rightarrow P \rightarrow X$, we have the *Atiyah sequence*[1] $0 \rightarrow \mathbf{L} \rightarrow \mathbf{Q} \rightarrow \mathbf{T}_X \rightarrow 0$ and the corresponding sheaf sequence

$$0 \rightarrow \mathfrak{L} \rightarrow Q \rightarrow \Theta \rightarrow 0. \tag{1}$$

We record some interpretations of the groups arising from the exact cohomology sequence of (1).

(i) $H^{\circ}(X,\mathfrak{L})$ represents the infinitesimal bundle automorphisms of P which project to the trivial automorphism of X.

(ii) $H^{\circ}(X,\mathfrak{Q})$ gives the infinitesimal bundle automorphisms of P.

(iii) $H^{\circ}(X, \Theta) \cong \mathfrak{a}$ represents the complex Lie algebra $\mathfrak{a}$ of A.

(iv) In the sequence $0 \rightarrow H^\circ(X,\mathcal{L}) \rightarrow H^\circ(X,\mathcal{Q}) \rightarrow H^\circ(X,\Theta)$, all homomorphisms are Lie algebra homomorphisms.

We now make one final observation. Let $\mathcal{S} \rightarrow X$ be a coherent free analytic sheaf, and let $B \subset A$ be a subgroup whose action on X lifts to action in $\mathcal{S}$. Then B acts on the cohomology groups $H^q(X,\mathcal{S})$ ($q = 0, 1, \ldots$) and these groups are in fact finite dimensional B-modules.

Definition 1: A bundle $\mathbf{E} \rightarrow X$ is *homogeneous* with respect to a complex subgroup $B \subseteq A$ if the action of B on X lifts to bundle action in $\mathbf{E}$.

This note is primarily concerned with some geometric interpretations of the representations defined above.

2. *Automorphisms and Deformations of Analytic Bundles.*—We shall consider *deformations*, as defined in reference 5, of the analytic bundle $G \rightarrow P \rightarrow X$; recall that the *infinitesimal tangents* to deformations are given as classes in $H^1(X,\mathcal{L})$. If we let $\mathcal{G}$ be the sheaf of germs of local holomorphic mappings of X into G, then the bundle P is uniquely prescribed by an element $\xi_P = \xi \in H^1(X,\mathcal{G})$. The Lie group A acts on $H^1(X,\mathcal{G})$, and an obvious class of deformations of $G \rightarrow P \rightarrow X$ is given by considering the bundles $a \cdot \xi \in H^1(X,\mathcal{G})$ ($a \in A$). These deformations are given infinitesimally as follows:

PROPOSITION 1. *Let δ: $H^\circ(X,\Theta) \rightarrow H^1(X,\mathcal{L})$ be the connecting homomorphism in the exact cohomology sequence of (1). Then the vectors in the subspace $Im\ \delta \subseteq H^1(X,\mathcal{L})$ are all tangent to deformations of P, and, in fact, for $\theta \in H^\circ(X,\Theta)$, $\delta(\theta)$ is tangent to the family of bundles $(exp\ t\ \theta) \cdot \xi$.*

Proof: The proof is a straightforward local calculation which we shall omit. However, we record three remarks:

(i) Ker $\delta \subseteq H^\circ(X,\Theta)$ is the complex subalgebra of a consisting of those infinitesimal automorphisms of X which lift to action in P.

(ii) If $G = \mathcal{C}^*(\mathcal{G} = \Omega^*)$, then $\mathcal{L} = \Omega$ and the deformations of P are locally parametrized by $H^1(X,\Omega)$. If X is Kählerian, then we have an exact sequence

$$0 \rightarrow H^1(\mathfrak{X},\mathbf{Z}) \xrightarrow{P} H^1(\mathfrak{X},\Omega) \rightarrow H^1(\mathfrak{X},\Omega^*) \xrightarrow{\delta^*} H^2(\mathfrak{X},\mathbf{Z}) \rightarrow 0 \qquad (2)$$

and $H^1(\mathfrak{X},\Omega)/pH^1(\mathfrak{X},\mathbf{Z}) = \mathcal{P}$ gives a global deformation space for *any* line bundle on X. ($\mathcal{P}$ is the *Picard variety* of X.)

LEMMA 1. *If X is Kählerian, then A acts trivially on the groups $H^p(X,\Omega^q)$.*

Proof: $H^r(X,\mathcal{C}) = \sum_{p+q=r} H^p(X,\Omega^q)$ (Ω^q = sheaf of holomorphic q-forms on X) and $H^p(X,\Omega^q) \cong H^{p,q}(X,\mathcal{C})$ under the Dolbeault isomorphism. If $\eta \in H^{p,q}(X,\mathcal{C})$, then η is a global (p, q)—form with $\partial\eta = 0 = \bar{\partial}\eta$. Let $\theta \in a$ and let $\mathcal{L}_\theta$ = Lie derivative along θ. Then $\mathcal{L}_\theta\eta = \partial i(\theta)\eta + i(\theta)\partial n = \partial i(\theta)\eta$. Thus, since X is Kählerian, $\mathcal{L}_\theta\eta \sim 0$ and this says that the infinitesimal representation of A on $H^{p,q}(X,\mathcal{C})$ is trivial. (What we have done essentially is to observe that the action of any $a \in A$ on X is homotopic to the identity.)

Definition 2: For $\xi \in H^1(X,\mathcal{G})$, we let $A_\xi \subseteq A$ be the connected complex Lie group with complex Lie algebra $a_\xi = \text{Ker}(\delta)$.

Remark: A_ξ is the largest subgroup of A with respect to which P is homogeneous.

We now write $\mathbf{L}(\mathbf{X})$ for $H^1(X,\Omega^*)$; $\mathbf{L}(\mathbf{X})$ is the abelian group of line bundles on X, and we shall write the composition law additively. Define a mapping $F: A \rightarrow \text{Hom}(\mathbf{L}(\mathbf{X}),\mathcal{P})$ as follows: for $a \in A$, $\xi \in \mathbf{L}(\mathbf{X})$, $F(a)\xi = a \cdot \xi - \xi$.

PROPOSITION 2. (i) *$F(a)\xi \in \mathcal{P}$ and $F(a)(\xi + \eta) = F(a)\xi + F(a)\eta$. (ii) If X is Kählerian, $F(ab)\xi = F(a)\xi + F(b)\xi$ for $a,b \in A$.*

Proof: In (2), $F(a)\xi \in H^1(X, \Omega^*)$ and $\delta(F(a)\xi) = \delta(a\cdot\xi - \xi) = a\cdot\delta(\xi) - \delta(\xi) = 0$. Also $a\cdot(\xi + \eta) = a\cdot\xi + a\cdot\eta$. Clearly (i) now follows.

If now X is Kählerian, then A acts trivially on $H^1(X,\Omega)$ (by the above remark). Hence, $F(a)F(b)\cdot\xi = 0$ for $a,b \in A$, $\xi \in \mathbf{L}(\mathbf{X})$. Thus, $F(a)\xi + F(b)\xi = F(a)F(b)\xi + F(a)\xi + F(b)\xi = F(a)(F(b)\xi + \xi) + F(b)\xi = ab\cdot\xi - b\cdot\xi + b\cdot\xi - \xi = F(ab)\xi$. *Q.E.D.*

Assume now that X is Kählerian. For fixed $\xi \in \mathbf{L}(\mathbf{X})$, define a homomorphism $F_\xi: A \to \mathcal{P}$ by $F_\xi(a) = F(a)\xi$. Then we clearly have:

PROPOSITION 3. *F_ξ is holomorphic and $Ker(F_\xi) = A_\xi$.*

Remark: The cohomology sequence of (1) for a line bundle $\xi \in \mathbf{L}(\mathbf{X})$ is

$$\to H^\circ(X,\mathcal{Q}_\xi) \to H^\circ(X,\Theta) \xrightarrow{\delta} H^1(X,\Omega) \to. \tag{3}$$

The mapping δ is the infinitesimal form of F_ξ, and Proposition 3 shows that δ is an algebra homomorphism when we consider $H^1(X,\Omega)$ as an abelian Lie algebra. This can be proved directly, and, as the proof may have some interest, we now give it.

PROPOSITION 4. *In (3) above, $\delta[\theta,\theta'] = 0$ for $\theta,\theta' \in H^\circ(X,\Theta)$, provided that X is Kählerian.*

Proof: Let $\omega \in H^1(X,\Omega^1)$ represent, in the Dolbeault sense, the characteristic class of $\xi_1 \in \mathbf{L}(\mathbf{X})$ ([1]). An easy calculation shows that $\delta(\theta) = i(\theta)\omega \in H^1(X,\Omega)$ where $\theta \in H^\circ(X,\Theta)$ and $i(\theta)\omega$ is the tensor contraction of ω by θ. Since A acts trivially on $H^1(X,\Omega)$, we then see that, if $\theta' \in H^\circ(X,\Theta)$, $L_{\theta'}(i(\theta)\omega) = 0$ in $H^1(X,\Omega)$, where $L_{\theta'}$ is the operation of taking the Lie derivative. (The operation $L_{\theta'}$ is the infinitesimal representation of A on $H^1(X,\Omega)$.) But $\delta[\theta,\theta'] = i([\theta,\theta'])\omega = L_\theta(i(\theta')\omega) - L_{\theta'}(i(\theta)\omega) = 0$ in $H^1(X,\Omega)$. *Q.E.D.*

3. *Equivariant Embeddings of Complex Manifolds.*—Let R, S be complex connected Lie groups, S a closed subgroup of R, such that the coset space $Y = R/S$ is a compact simply connected algebraic variety. Let $f: X \to Y$ be a holomorphic mapping and $\sigma: B \to R$ a holomorphic homomorphism for some complex subgroup $B \subseteq A$.

Definition 3: f is equivariant with respect to B if, for any $x \in X$, $b \in B$, $f(b\cdot x) = \sigma(b)\cdot f(x)$.

If the bundle $\mathbf{E} \to X$ has the property that the global sections $(H^\circ(X,\mathcal{E}))$ generate the fibre $\mathbf{E}_x$ for each $x \in X$ (that is to say, the linear system of the bundle $\mathbf{E}$ has no base points), then there is classically defined a mapping $f_E = f: X \to Y$ where $Y = R/S$ is a Grassmann variety.

THEOREM. *The mapping f is equivariant with respect to $A_\xi(\mathbf{E} = \xi)$ where $\sigma: A_\xi \to R = GL(H^\circ(X,\mathcal{E}))$ is the induced representation on sheaf cohomology. If f is an embedding, then σ is faithful, and the automorphism group A_ξ is induced from those automorphisms acting in Y which leave $f(X)$ invariant. The group A_ξ is maximal with respect to this property, and the normal bundle $\mathbf{N}_f$ to $f(X)$ in Y is homogeneous with respect to A_ξ.*

The proof of this theorem is again fairly straightforward from the definitions, and, rather than go through the proof in detail, we shall describe some geometric corollaries.

4. *Some Applications.*

(i) COROLLARY 1. *If the vector bundle* $\mathbf{E} \to X$ *is ample and if* $H^1(X,\mathcal{L}) = 0$ *(rigidity of P), then the automorphisms A of X are all induced by motions in the ambient space relative to the embedding defined by* $H^\circ(X,\mathcal{E})$.

(ii) COROLLARY 2. *If* $X \subset P_N$ *is an algebraic variety in projective N-space, and if* $\xi \in \mathbf{L}(\mathbf{X})$ *is the line bundle of a linear hyperplane section, then the group* $A_\xi \subset A$ *is faithfully represented as the largest subgroup of PG(N) (= projective group in N + 1 variables) leaving X fixed. If X is regular, then* $A_\xi = A$. *In general there is an exact sequence of complex Lie groups* $1 \to A_\xi \to A \xrightarrow{p_\xi} T_\xi \to 1$, *where* T_ξ *is a connected analytic group of translations of the Picard variety of X with Lie algebra* $\delta(H^\circ(X,\Theta))$ *in (3). (Compare with reference 2, §3.)*

(iii) Let X be a regular algebraic variety, and let $f:X \to P_N$ be a projective embedding with normal bundle $\mathbf{N}_f$. If $H^1(X,\mathfrak{N}_f) = 0$, then the Theorem on Completeness of the Characteristic System given in reference 4 tells us that a neighborhood of the origin in $H^\circ(X,\mathfrak{N}_f)$ parametrizes the local deformations of $f(X)$ in P_N. The bundle $\mathbf{N}_f$ is defined by the exact sequence

$$0 \to \mathbf{T}_{f(X)} \to \mathbf{T}_{P_N}|f(X) \to \mathbf{N}_f \to 0,$$

and, if

$$\mathfrak{s} = H^\circ(P_N,\Theta_{P_N}),$$

we have a diagram

$$\begin{array}{ccccccccc} 0 \to & H^\circ(X,\Theta_X) & \to & H^\circ(X,\Theta_{P_N}|X) & \to & H^\circ(X,\mathfrak{N}_f) & \xrightarrow{\delta} & H^1(X,\Theta_X) \to & (4) \\ & \| & & \uparrow & \overset{\mu}{\nearrow} & & & & \\ 0 \to & \mathfrak{a} & \to & \mathfrak{s} & & & & & \end{array}$$

Clearly, $\mu:\mathfrak{s}/\mathfrak{a} \to H^\circ(X,\mathfrak{N}_f)$ is an injection for suitable f, and $\delta\cdot\mu = 0$. This says geometrically that the coset space S/A *effectively* parametrizes locally some of the variations of $f(X)$ in P_N.

Now let $\eta \in H^\circ(X, \mathfrak{N}_f)$ and let $X^t = (|t| < \epsilon, X_0 = f(X))$ be a deformation of $f(X)$ in P_N with tangent η. Since A acts on $\mathbf{N}_f$, it is represented by a homomorphism $\sigma:A \to GL(H^\circ(X,\mathfrak{N}_f))$. On the subspace $\mathfrak{s}/\mathfrak{a}$ of $H^\circ(X,\mathfrak{N}_f)$, σ is just the adjoint representation of $\mathfrak{a}$ on $\mathfrak{s}/\mathfrak{a}$. We have

PROPOSITION 5. *In order that the subgroup* $A \subseteq S$ *act on each of the submanifolds* X_t, *it is necessary that* $\sigma(a)\eta = \eta$ *for all* $a \in A$. *In particular, if* $\eta \in \mathfrak{s}/\mathfrak{a}$, *then A does not act on the manifolds* X_t.

Remarks: The proof is again by a straightforward calculation. This result may be used to give an extrinsic geometric proof of the rigidity of a class of algebraic varieties, including all rational homogeneous varieties. This will be carried out in reference 3.

[1] Atiyah, M. F., "Complex analytic connexions in fibre bundles," *Trans. Amer. Math. Soc.*, 85, 181–207 (1957).

[2] Blanchard, A., "Sur les variétés analytiques complexes," *Ann. école normale supérieure*, 73, 157–202 (1956).

[3] Griffiths, P. A., "On certain homogeneous complex manifolds," to appear in *Acta Math.*

[4] Kodaira, K., "A theorem of completeness of characteristic systems for analytic families of compact submanifolds," to appear in *Ann. of Math.*

[5] Kodaira, K., and D. C. Spencer, "On deformations of complex structures I, II," *Ann. of Math.*, 67, 328–466 (1958).