A Note on the Spectrum of Cusp Forms for Congruence Subgroups

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For a fixed Riemannian manifold the asymptotics of the eigenvalues of the corresponding Laplace-Beltrami operator is undertstood in great generality and is known as Weyl's law. In this note we examine the distribution of eigenvalues in a fixed interval over a particular sequence of surfaces. Let H be the Lobachevsky upper half plane with its noneuclidian metric, and let Δ be the corresponding Laplacian. Denote by $\Gamma(1)$ the modular group $PSL(2,Z)$ and by $\Gamma(N)$ its principal congruence subgroup of level N. Let the index $[\Gamma(1),\Gamma(N)]$ be denoted by $\mu(N)$, and let $\nu(N)$ be the number of inequivalent cusps of $H/\Gamma(N)$. Then it is well known that

$$\mu(N) = \frac{N^3}{2} \prod_{p|N} (1-p^{-2}) \quad \text{if} \quad N > 2$$

$$\nu(N) = \mu(N) / N$$

See for example Shimura [6].

Let $\lambda_{j,N}$, $j = 0,1,\ldots$ be the discrete spectrum of Δ for $H/\Gamma(N)$, see [4]. Thus $0 = \lambda_{0,N} < \lambda_{1,N} \leq \lambda_{2,N} \cdots$, and it is known Selberg [5] that $\lambda_{1,N} \geq 3/16$. It is also known that the Eisenstein series for $\Gamma(N)$ have no poles in $[1/2,1)$ so that $\lambda_{j,N}$, $j \geq 1$ correspond to Maass cusp forms. As is customary we denote by $r_{j,N}$ the quantities $\sqrt{\lambda_{j,N}-1/4}$, which is a more convenient parametrization of the spectrum. Thus $r_{j,N}$ is in the union of the two segments $[0,\infty)$ and $[0,i/2]$ which we denote by C.

We are interested in the distribution of the $r_{j,N}$ as $N \to \infty$. This can be thought of as the "N aspect" of Weyl's law, and this relationship is similar to the K-t (conductor-height) relationship for L functions and the zeta function, see [7]. In fact we will make use of Siegel's theorem [7] concerning zeros of cyclotomic zeta functions. For $[\alpha,\beta]$ a subinterval of C we let $M(\alpha,\beta,N) = \#\{j: r_{j,N} \in [\alpha,\beta]\}$.

THEOREM. As $N \to \infty$ we have

$$\text{(i)} \quad \frac{M(\alpha,\beta,N)}{\mu(N)} \to \frac{1}{6}\int_\alpha^\beta r\left(\frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}}\right) dr \quad \text{if} \quad [\alpha,\beta] \subset [0,\infty)$$

and

$$\text{(ii)} \quad \frac{M(\alpha,\beta,N)}{\mu(N)} \to 0 \qquad \text{if} \quad [\alpha,\beta] \subset [0,i/2]$$

This theorem says that the $r_{j,N}$'s distribute themselves uniformly in C with respect to the measure

$$d_m = \begin{cases} 0 & \text{on } [0,i/2] \\ \frac{r}{6}\tanh \pi r \, dr & \text{on } [0,\infty) \end{cases} .$$

This result should be compared with the results of of Iwaniec and Szmidt [3], which concern small eigenvalues for $\Gamma_0(N)$, and where the question of the behavior of $r_{j,N}$ for eigenvalues inside the continuous spectrum is raised. The above theorem should also be compared with the recent work of Huber [2] concerning compact Riemann surfaces of constant curvature -1. In fact our method gives a simple proof of the theorem of Huber which states

THEOREM A (Huber). Let F_N be a sequence of such compact surfaces, and let $r_{j,N}$ be as above. Then if $\ell(F_N)$ tends to infinity, where (F_N) is the length of the shortest closed geodesic on F_N then (i) and (ii) above hold with $\mu(N) = \frac{3}{\pi}\,\mathrm{VOL}(F_N)$.

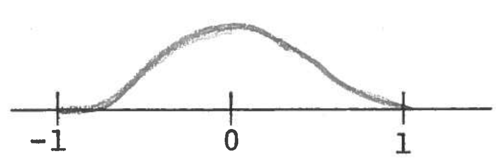
To prove the theorem we need some lemmata.

LEMMA 1. Let $T > 0$ be fixed, and let $\chi_{[-T,T]}(x)$ be the characteristic function of the interval $[-T,T]$, then for $\varepsilon > 0$ given there are even functions h_1, h_2 whose Fourier transforms are of compact support and for which

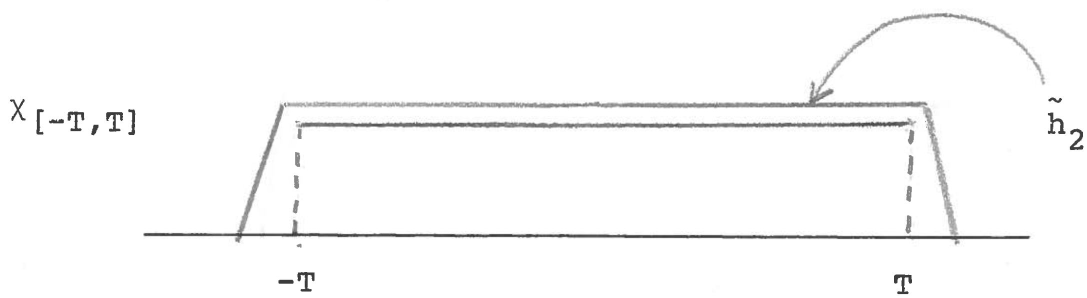
(a) $$h_1 \le \chi_{[-T,T]} \le h_2$$

(b) $$\int_{-\infty}^{\infty} (h_2 - h_1)(1 + x^2)\,dx < \varepsilon .$$

<u>Proof</u>: Let $\hat{\phi}(\xi)$ be a positive even function with support in $[-1,1]$, with $\hat{\phi}(0) = 1$ and for which $\phi \ge 0$

$\phi =$ 

For the construction of h_2, we first construct $\tilde{h}_2$ as shown



so that

$$\| h_2 - \chi_{[-T,T]} \|_{1,dP} = \int_{-\infty}^{\infty} |h_2 - \chi_{[-T,T]}|(x)(x^2+1)\, dx < \frac{\varepsilon}{4}$$

i.e. $dP = (x^2 + 1)\, dx$.

If $\phi(x) = \frac{1}{\varepsilon}\phi(x/\varepsilon)$ then $\phi_\varepsilon * \tilde{h}_2 \to \tilde{h}_2$ in both $L^1(R,dP)$ and $L^\infty(R)$. Being the convolution of positive functions it follows that for ε small enough

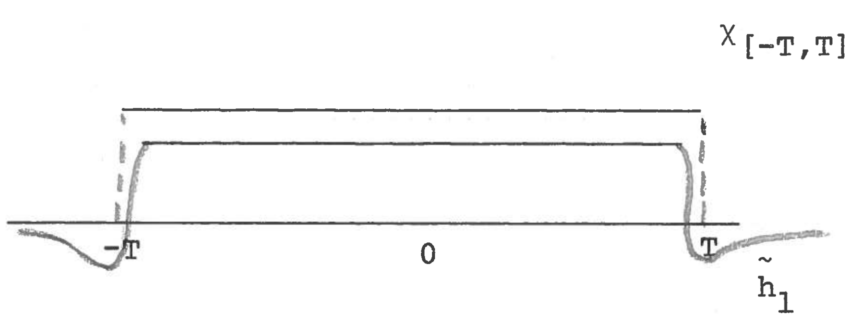
$$h_2 = \phi_\varepsilon * \tilde{h}_2$$

satisfies

$$h_2 \geq \chi_{[-T,T]} \quad \text{and} \quad \| h_2 - \chi_{[-T,T]} \|_{1,dP} < \frac{\varepsilon}{2}$$

and clearly $\hat{h}_2$ is of compact support.

For the construction of h_1 we first let $\tilde{h}_1$ be as shown.



$\tilde{h}_1$ can also be chosen so that

$$\|\chi_{[-T,T]} - \tilde{h}_1\|_{1,dP} < \frac{\varepsilon}{2}$$

and asymptotically for large $|x|$

(1) $$\tilde{h}_1(x) \leq -\frac{c}{|x|^5} \quad \text{for an absolute positive } c.$$

Again as $\varepsilon \to 0$, $\phi_\varepsilon * \tilde{h}_1 \to \tilde{h}_1$ in $L^1(R,dP)$, as well as in $L^\infty(R)$. It is also clear that, since we have chosen $\tilde{h}_1$ satisfying (1) and since ϕ is rapidly decaying in $|x|$, that for $|x|$ large, depending only on T, and uniformly for ε small, we have $(\phi_\varepsilon * h)(x) < 0$. By these remarks it follows that for ε small enough

$$h_1 = \phi_\varepsilon * \tilde{h}_1 \leq \chi_{[-T,T]}$$

and

$$\|\chi_{[-T,T]} - h_1\|_{1,dP} < \frac{\varepsilon}{2},$$

so that h_1 satisfies the lemma.

<u>Lemma 2</u>. If $\ell(N)$ is the "neck" of $\Gamma(N)$, that is the length of the shortest closed geodesic on $H/\Gamma(N)$ then $\Gamma(N) \to \infty$ as $N \to \infty$.

<u>Proof</u>: $\Gamma(N)$, $N \geq 2$ is free. We must show that the trace of any $\gamma \in \Gamma(N)$, $\gamma \neq$ parabolic tends to $\pm\infty$ as $N \to \infty$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$, $\text{Trace}(\gamma) \equiv 2 \pmod{N}$, so if $\text{Trace}(\gamma) \neq 2$, i.e. γ is not parabolic then $|\text{Trace}(\gamma)| \geq |N|-2$.

Denote by $\phi_N(s)$ the determinant of the Eisenstein series matrix $\phi_{ij}(s)$, for $\Gamma(N)$. (See Kubota [ ] for the definitions.) Let $\phi(N)$ be the usual Euler phi function and let $\chi_1, \chi_2, \ldots, \chi_{(N)}$ be the Dirichlet characters to modulus N. Let $L(\chi,s)$ be the L-function

$$L(\chi,s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Lemma 3. The poles of $\phi_N(s)$ are contained within the nontrivial zeros of the function

$$(L(x_1,2s)\ L(x_2,2s)\ \ldots\ L(\chi_{\phi(N)},2s))^{\nu(N)/\phi(N)} \triangleq (\zeta_N(2s))^{\nu(N)/\phi(N)}$$

where we count zeros and poles with multiplicity.

Proof: See Efrat-Sarnak [1].

Lemma 4. Let h be a fixed rapidly decreasing even function on R, then

$$\int_{-\infty}^{\infty} \left(\frac{\phi_N'}{\phi_N}\right) \left(\frac{1}{2} + ir\right) h(r)\, dr = o(\mu(N)) \quad \text{as } N \to \infty.$$

Proof: In what follows << or 0 will mean the implied constants depend only on h. Denote the poles of ϕ_N in $\mathrm{Re}(s) < 1/2$ by $\phi' = \beta' + i\gamma'$. Using the product factorization of $\phi_N(s)$ the left hand side of Lemma 4 is up to bounded quantities, given by H, where

$$H = \int_{-\infty}^{\infty} \sum_{\rho'} \frac{1 - 2\beta'}{(\frac{1}{2} - \beta')^2 + (t-\gamma')^2} h(t)\, dt$$

$$<< \int_{-\infty}^{\infty} \sum_{\rho'} \frac{1 - 2\beta'}{(\frac{1}{2} - \beta')^2 + (t-\gamma')^2} |h(t)|\, dt$$

which by Lemma 3 is

$$\leq \frac{\nu(N)}{\phi(N)} \int_{-\infty}^{\infty} \sum_{\rho} \frac{(1-2\beta)}{(\frac{1}{2} - \beta) + (1-\gamma)^2} |h(t)|\, dt$$

where the ρ's are the nontrivial zeros of $\zeta_N(2s)$. Thus

$$H << \frac{\nu(N)}{\phi(N)} \sum_{\rho} \int_{-\infty}^{\infty} \frac{h\left((\frac{1}{2} - \beta)u + \gamma\right)}{1+u^2}\, du$$

$$<< \frac{\nu(N)}{\phi(N)} \left\{ \sum_{|\gamma| \leq 5} 1 + 2 \sum_{\gamma \geq 5} \int_{-\infty}^{\infty} |h|\,[(\frac{1}{2} - \beta)u + \gamma]\, \frac{du}{1+u^2} \right\} \qquad (2)$$

To estimate a term in the sum $\gamma \geq 5$, we split it into

$$\int_{\frac{-\gamma - \gamma^{1/2}}{\frac{1}{2} - \beta}}^{\frac{-\gamma + \gamma^{1/2}}{\frac{1}{2} - \beta}} |h[(\frac{1}{2} - \beta)u + \gamma]|\, \frac{du}{u^2+1} + \int_{|(\frac{1}{2} - \beta)u + \gamma| > \gamma^{1/2}} h[(\frac{1}{2} - \beta)u + \gamma]\, \frac{du}{1+u^2}$$

The first term is

$$<< \left(\frac{\gamma - \gamma^{1/2}}{\frac{1}{2} - \beta}\right)^{-1} - \left(\frac{\gamma + \gamma^{1/2}}{\frac{1}{2} - \beta}\right)^{-1} \leq \frac{2}{\gamma^{1/2}(\gamma - 1)}$$

The second is

$$<< \gamma^{-2} \int_{-\infty}^{\infty} \frac{du}{1+u^2}$$

Thus

$$H << \frac{\nu(N)}{\phi(N)} \left\{ \sum_{|\gamma| \leq 5} 1 + \sum_{|\gamma|>5} |\gamma|^{-3/2} \right\}$$

Now using standard methods to estimate the number of zeros of $\zeta_N(s)$ in height T, see Siegel [7] we obtain

$$\# \{\gamma : |\gamma| \leq T\} << T \log T \, \phi(N) \log N$$

whence

$$H << \frac{\nu(N)}{\phi(N)} \cdot \phi(N) \log N = \nu(N) \log N = o(\mu(N)) .$$

Before turning to the proof of the theorem we need Selberg's formula for the groups $\Gamma(N)$. If g is even, smooth and of compact support in R, and if N is large enough so that support $g \subset [-\ell(N), \ell(N)]$, then the hyperbolic terms in the formula drop out and there are no elliptic terms so that the formula reads (see Kubota [4])

$$\sum_{\pm j} h(r_{j,N}) = \frac{\mu(N)}{6} \int_{-\infty}^{\infty} r \tanh(\pi r)\, h(r)\, dr$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\phi_N'}{\phi_N} (\tfrac{1}{2} + ir)\, dr - \frac{\nu(N)}{\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(1+ir)\, dr$$

$$- 2\nu(N)(\log 2) g(0) + \frac{1}{2} \left(\nu(N) - \text{TRACE } \Phi(\tfrac{1}{2})\right) h(0)$$

where

$$h = \hat{g} . \qquad (3)$$

<u>Proof of the Theorem:</u> We begin with

$$M(0,i/2,N) = o(\mu(N)) .$$

Choose g even compact support, positive so that $h \geq 0$ and $h(0) = 1$. This can easily be done. It follows that

$$h(iu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) \cosh(ux)\, dx \geq 1 \quad \text{for} \quad u \in R$$

Thus

$$M(0,i/2,N) \leq \sum_j h(r_{j,N}) \qquad (4)$$

Now for N large enough it follows by Lemma 2 that $\operatorname{supp} g \subset [-\ell(N),\ell(N)]$ so that we may apply (3). By Lemma 4 the second term on the right of (3) is $o(\mu(N))$, the last three are obviously $o(\mu(N))$, thus (4) yields

$$M(0,i/2,N) \leq \frac{\mu(N)}{12} \int_{-\infty}^{\infty} r \tanh(\pi r)\, h(r)\, dr + o(\mu(N)) .$$

And so

$$\overline{\lim_{N\to\infty}} \frac{M(0,i/2,N)}{\mu(N)} \leq \frac{1}{12} \int_{-\infty}^{\infty} r \tanh(\pi r)\, h(r)\, dr$$

but this is true for any h for which g is of compact support. Thus

$$\overline{\lim_{N\to\infty}} \frac{M(0,i/2,N)}{\mu(N)} = 0 .$$

To prove (i) it clearly suffices to prove that

$$M(0,T,N) \sim \frac{\mu(N)}{6} \int_0^T r \tanh(\pi r)\, dr \ , \quad \text{for every } T > 0.$$

For any $\varepsilon > 0$ we find h_1, h_2 as in Lemma 1. For N large enough formula (3) applies, all terms besides $\sum_{r_j,\ r_j \text{ real}} h(r_{j,N})$ and $\frac{\mu(N)\pi}{12} \int \ldots$ are $o(\mu(N))$ as before (the $\sum_{r_j \text{ complex}}$ is $o(\mu(N))$ by (ii) above).

Thus by the properties of h_1 and h_2

$$\frac{1}{12} \int_{-\infty}^{\infty} r \tanh(\pi r)\, h_1(r)\, dr + o(1) \le \frac{M(0,T,N)}{\mu(N)}$$

$$\le \frac{1}{12} \int_{-\infty}^{\infty} r \tanh\,(\pi r)\, h_2(r)\, dr + o(1)$$

Therefore

$$\frac{1}{12} \int_{-\infty}^{\infty} r \tanh(\pi r)\, h_1(r)\, dr \le \underline{\lim}\, \frac{M(0,T,N)}{\mu(N)}$$

$$\le \overline{\lim}\, \frac{M(0,T,N)}{\mu(N)} \qquad \le \frac{1}{12} \int_{-\infty}^{\infty} r \tanh(\pi r)\, h_2(r)\, dr$$

But by Lemma 1

$$\| h_j - \chi_{[-T,T]} \|_{1,dP} < \varepsilon$$

for $j = 1,2$, where ε is arbitrarily small; it follows that

$$\lim_{N\to\infty} \frac{M(0,T,N)}{\mu(N)} = \frac{1}{12} \int_{-T}^{T} r \tanh(\pi r)\, dr \ ,$$

which completes the proof of the theorem.

References

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