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A TRANSCENDENTAL METHOD IN ALGEBRAIC GEOMETRY

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1. Introduction and an example from curves.

It is well known that the basic objects of algebraic geometry, the smooth projective varieties, depend continuously on parameters as well as having the usual discrete invariants such as homotopy and homology groups. What I shall attempt here is to outline a procedure for measuring this continuous variation of structure. This method uses the periods of suitably defined rational differential forms to construct an intrinsic " continuous " invariant of arbitrary smooth projective varieties. The original aim in defining this " period matrix " of an algebraic variety was to give, at least in some cases, a complete invariant (i. e. " moduli ") of the algebraic structure, as turns out to happen for curves. It is too soon to evaluate the success of this program, but a few interesting things have turned up, and there remain very many attractive unsolved problems. In presenting this talk, I shall not give references as these, together with a more detailed discussion of the material discussed, may be found in my survey paper which appeared in the March (1970) *Bulletin of the American Mathematical Society.*

Let me begin by discussing the example of hyperelliptic curves. Consider the family of affine curves with the equation

$$y^2 = (x - s_1) \dots (x - s_{2g+2}).$$

Denoting by V_s the complete curve corresponding to $s = (s_1, \dots, s_{2g+2})$ and letting

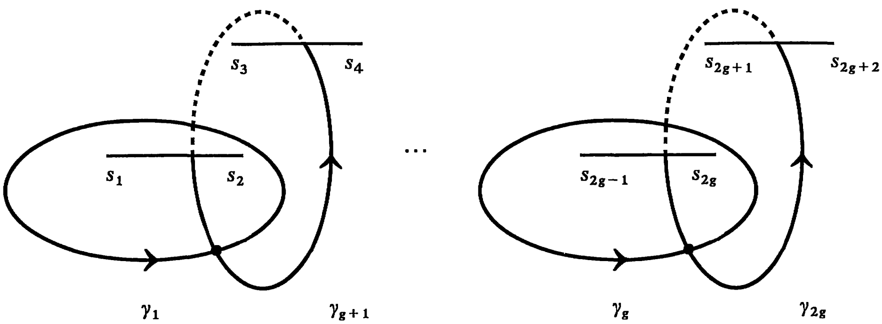
$$S = \{ s : \prod_{j<k} (s_j - s_k) \neq 0 \},$$

we see that $\{ V_s \}_{s \in S}$ forms an algebraic family of non-singular curves of genus g. Furthermore, for a suitable *smooth completion* $\bar{S}$ of S (e. g. $\bar{S} = \mathbb{P}_{2g+2}$), we may enlarge our family to $\{ V_s \}_{s \in \bar{S}}$ by adding suitable degenerate curves $V_{\bar{s}}$ corresponding to the points $\bar{s} \in \bar{S} - S$. The notations $\{ V_s \}_{s \in S}$ and $\{ V_s \}_{s \in \bar{S}}$ will be used throughout this talk to represent respectively an algebraic family of smooth, projective varieties V_s with smooth parameter space S, and a completion of this family where $\bar{S}$ is smooth and $\bar{S} - S = D_1 \cup \dots \cup D_l$ is a divisor with normal crossings. The varieties $V_{\bar{s}}$ ($\bar{s} \in D_j$) may be thought of as singular specializations of the general V_s.

On the curve V_s we consider a basis $\varphi_1, \dots, \varphi_g$ for the holomorphic differentials and a *canonical basis* $\gamma_1, \dots, \gamma_{2g}$ for the first homology $H_1(V_s, \mathbb{Z})$. Thus we might choose

$$\varphi_\alpha = \frac{x^{\alpha-1}dx}{y} \qquad (\alpha = 1, \dots, g)$$

and, upon representing V_s as a 2-sheeted covering of the x-line, we have the picture



The choice of the $\{\varphi_\alpha\}$ is determined up to a substitution $\varphi_\alpha \to \sum_{\beta=1}^{g} A_\alpha^\beta \varphi_\beta$, $\det(A_\alpha^\beta) \neq 0$, and the $\{\gamma_\rho\}$ are determined up to a transformation $\gamma_\rho \to \sum_{\sigma=1}^{2g} T_\rho^\sigma \gamma_\sigma$ where $T = (T_\rho^\sigma)$ is a $2g \times 2g$ integral matrix which preserves the intersection matrix $Q = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ of the $\{\gamma_\rho\}$. Thus $A \in GL(g, \mathbb{C})$ and $T \in Sp(g, \mathbb{Z})$.

We now form the period matrix

$$\Omega(s) = \underbrace{\begin{pmatrix} \int_{\gamma_1} \varphi_1 & \cdots & \int_{\gamma_{2g}} \varphi_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \varphi_g & \cdots & \int_{\gamma_{2g}} \varphi_g \end{pmatrix}}_{2g} \Bigg\} g,$$

which is determined up to the equivalence relation

$$\Omega \sim A \Omega T$$

arising from the indeterminacy of the $\{\varphi_\alpha\}$ and $\{\gamma_\rho\}$. Because of the obvious relations

$$\begin{cases} \int_{V_s} \varphi_\alpha \wedge \varphi_\beta = 0 \\ \sqrt{-1} \int_{V_s} \varphi_\alpha \wedge \bar{\varphi}_\alpha > 0, \end{cases}$$

the period matrix $\Omega(s)$ satisfies the *Riemann bilinear relations*

$$\begin{cases} \Omega Q {}^t\Omega = 0 \\ \sqrt{-1}\Omega Q {}^t\bar{\Omega} > 0. \end{cases}$$

Thus, if we let D be the set of all $g \times 2g$ matrices Ω which satisfy the Riemann bilinear relations and with the equivalence $\Omega \sim A\Omega$ ($A \in GL(g, \mathbb{C})$), we see that the periods of the holomorphic differentials on V_s lead to the period mapping

$$\Omega: \quad S \rightarrow D/Sp(g, \mathbb{Z}),$$

where $Sp(g, \mathbb{Z})$ acts on D by sending Ω into $\Omega {}^t T^{-1}$. We recall that D is a complex manifold which is biholomorphic to the Siegel-upper-half-plane of all $g \times g$ matrices $Z = X + \sqrt{-1}Y$ with $Z = {}^tZ$, $Y > 0$. Furthermore, D is a homogeneous complex manifold with automorphism group $Sp(g, \mathbb{R})$ which acts in the same way as $Sp(g, \mathbb{Z})$ above. For $g = 1$, D is of course the usual upper half plane.

Here are a few properties of the period mapping:

(*a*) The point $\Omega(s)$ depends only on the intrinsic structure of V_s. Furthermore, $\Omega(s) = \Omega(s')$ if, and only if, the curves V_s and $V_{s'}$ are isomorphic (Torelli's theorem). Thus the period matrix gives a complete invariant for non-singular curves.

To discuss the next two properties, we need to digress a little about the monodromy group of a family of smooth algebraic varieties. In the case of our family of hyperelliptic curves, the canonical basis $\{ \gamma_\rho \}$ of $H_1(V_s, \mathbb{Z})$ will change when we displace V_s around a closed path in the parameter space S. More precisely, fixing a base point $s_0 \in S$ and letting $V = V_{s_0}$, the fundamental group $\pi_1(S)$ acts on the homology $H_1(V, \mathbb{Z})$. As is always the case, this action preserves the intersection pairing on homology, and we have then the monodromy representation

$$\begin{array}{c} \pi_1(S) \overset{\rho}{\dashrightarrow} \text{Aut}\,(H_1(V, \mathbb{Z})) \rightarrow \text{Aut}\,(D) \\ \qquad\qquad \| \wr \\ \qquad\qquad Sp(g, \mathbb{Z}) \end{array}$$

The image $\Gamma = \rho(\pi_1(S))$ will be called the *monodromy group.*

(*b*) For $g = 1$, the monodromy group is of finite index in $SL(2, \mathbb{Z}) \cong Sp(1, \mathbb{Z})$ (For an arbitrary family of elliptic curves, Γ is either a finite group or is of finite index in $SL(2, \mathbb{Z})$). This result should be interpreted as being a first suggestion that the monodromy group in an algebraic family of algebraic varieties has extremely remarkable properties.

(*c*) A further indication of this is the " rigidity property ", due to Grothendieck in this case. This states that if we have two families of curves $\{ V_s \}_{s \in S}$, $\{ V'_s \}_{s \in S}$ with the same parameter space S, with $V_{s_0} = V'_{s_0}$, and with the same monodromy representations ρ and ρ', then the period mappings Ω and Ω' are the same. In other words, the period mapping is determined by the monodromy representation plus its value at one point.

(*d*) The next property may perhaps be thought of as relating algebraic geometry to group representations. We recall that the study of the discrete series representations of the automorphism group $Sp(g, \mathbb{R})$ is intimately related to the construction of certain Γ-invariant meromorphic functions on D. If ψ is one such *automorphic function*, then the composite

$$\psi \circ \Omega$$

turns out to be a *rational* function on S. Roughly speaking, we may say that the study of $L^2(Sp(g, \mathbb{R}))$ leads to functions which uniformize the period mapping (" automorphic function property ").

The proofs of properties (*b*), (*c*), (*d*) above may be based on studying asymptotically the period matrix $\Omega(s)$ as s tends to a point $\bar{s} \in \bar{S} - S$. More precisely, a neighborhood in S of a point $s \in S - S$ will be a *punctured polycylinder*

$$P^* \cong \underbrace{\Delta^* \times \ldots \times \Delta^*}_{k} \times \underbrace{\Delta \times \ldots \times \Delta}_{m-k}$$

where Δ is a unit disc in $\mathbb{C}$, $\Delta^* = \Delta - \{0\}$ is the punctured disc, and $\dim S = m$. By localizing the period mapping at infinity, we will have a holomorphic mapping

$$\Omega: \quad P^* \to D/\Gamma$$

where we are interested in the behavior of $\Omega(s)$ as $\|s\| \to 0$ $(s = (s_1, \ldots, s_m) \in P^*)$. This asymptotic analysis of the period mapping is a purely function-theoretic problem which, in the end, should provide the best general method for proving the various global properties of Ω including the analogues of (*b*)-(*d*) above.

2. Construction and elementary properties of the period mapping.

We first observe that giving a $g \times 2g$ matrix Ω with the condition rank $(\Omega) = g$ and the equivalence relation $\Omega \sim A\Omega$ $(A \in GL(g, \mathbb{C}))$ is the same as giving a point $\Omega \in G(g, 2g)$, the Grassmann variety of g-planes in $\mathbb{C}^{2g}$. In fact, the point Ω is the point in $\mathbb{C}^{2g}$ spanned by the row vectors of the matrix Ω. Thus, giving the period matrix $\Omega(s)$ above is the same as giving a g-dimensional subspace of $H^1(V, \mathbb{C})$, this subspace being determined up to the monodromy group Γ. It is now easy to see that this g-dimensional subspace is simply the g-plane

$$H^{1,0}(V_s) \subset H^1(V_s, \mathbb{C})$$

spanned by the holomorphic 1-forms, followed by the identification

$$H^1(V_s, \mathbb{C}) \cong H^1(V, \mathbb{C})$$

which is determined up to Γ. Thus, giving the period matrix $\Omega(s)$ is equivalent to giving the g-dimensional subspace $H^{1,0}(V_s, \mathbb{C})$ of $H^1(V, \mathbb{C})$, and both of these are determined up to the monodromy group.

In general, let $\{V_s\}_{s \in S}$ be a family of smooth, projective algebraic varieties, and introduce the notations, $E = H^n(V_{s_0}, \mathbb{C})$, $E_{\mathbb{R}} = H^n(V_{s_0}, \mathbb{R})$, $E_{\mathbb{Z}} = H^n(V_{s_0}, \mathbb{Z})$. Using standard Kähler manifold theory we find that the cup product on $H^*(V, \mathbb{C})$ together with the Kähler class of the projective embedding give rise to a non-degenerate bilinear form

$$Q: \quad E \otimes E \to \mathbb{C}$$

which is rational on $E_{\mathbb{Z}}$, is invariant under the monodromy group Γ, and satisfies $Q(e, e') = (-1)^n Q(e', e)$. We will denote by G, $G_{\mathbb{R}}$, $G_{\mathbb{Z}}$ respectively the automorphism groups of E, $E_{\mathbb{R}}$, $E_{\mathbb{Z}}$ which preserve the bilinear form Q. $G_{\mathbb{C}}$ is a complex semi-simple algebraic group, $G_{\mathbb{R}}$ is a real form of $G_{\mathbb{C}}$, and $G_{\mathbb{Z}}$ is an arithmetic subgroup of $G_{\mathbb{R}}$ such that the monodromy group $\Gamma \subset G_{\mathbb{Z}}$.

From Hodge theory we recall the *Hodge decomposition*

$$H^n(V_s, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(V_s) \qquad (H^{p,q}(V_s) = \overline{H^{q,p}(V_s)}),$$

and using this we define the *Hodge filtration* $F^0(V_s) \subset \ldots \subset F^n(V_s) = H^n(V_s, \mathbb{C})$ by the formula

$$F^p(V_s) = H^{n,0}(V_s) + \ldots + H^{n-p,p}(V_s).$$

Using the Kodaira-Spencer continuity theorem, it follows that $F^p(V_s)$ is a continuously varying subspace of $H^n(V_s, \mathbb{C})$. Consequently, if we identify all $H^n(V_s, \mathbb{C})$ with $E = H^n(V_{s_0}, \mathbb{C})$ and let $F(E)$ be the *flag manifold* of all filtrations $F^0 \subset \ldots \subset F^n = E$, $\dim F^p = \dim F^p(V_s)$, then we have a continuous mapping

$$\Omega: \; S \to F(E)/\Gamma$$

which is the first form of the general period mapping. It will be convenient to write $\Omega(s) = (\Omega^0(s), \ldots, \Omega^n(s))$ where the $\Omega^p(s)$ are subspaces of $F(E)$ taken modulo Γ. Using the structure equations of the Kodaira-Spencer-Kuranishi theory of deformation of complex structure, it follows that $\Omega(s)$ varies holomorphically with $s \in S$.

The period mapping Ω will satisfy three bilinear relations, two of which are classical and generalize the Riemann-bilinear relations, and one which is non-classical but which is crucial for understanding the general period mapping. Recalling the bilinear form Q mentioned above, these bilinear relations are

(I) $\quad Q(\Omega^p, \Omega^{n-p-1}) = 0$
(II) $\quad (\sqrt{-1})^n Q(\Omega^p, \bar{\Omega}^p) > 0$ } *Hodge-Riemann bilinear relations*
(III) $\quad Q(d\Omega^p, \Omega^{n-p-2}) = 0$ *infinitesimal bilinear relation.*

The first relation is self-explanatory; the second means that, for any choice of basis $\{e_\alpha\}$ for Ω^p, the Hermitian matrix

$$(\sqrt{-1})^n Q(e_\alpha, e_\beta)$$

is non-singular and has a fixed signature; and the third bilinear relation means that

$$Q\left(\frac{\partial}{\partial s_j}\{\Omega^p(s)\}, \; \Omega^{n-p-2}(s)\right) = 0$$

where $(s_1, \ldots, s_n)$ are local coordinates on S.

Suppose now that we let $\check{D}$ be the algebraic variety of all points $(F^0, \ldots, F^n) \in F(E)$ which satisfy (I), and let D be the open set in $\check{D}$ of all points which satisfy (II). Then $\check{D}$ is acted on transitively by the group $G_{\mathbb{C}}$, and D turns out to be the $G_{\mathbb{R}}$ orbit of a suitable point in $\check{D}$. Thus we have a diagram

$$\begin{array}{ccc} D & \subset & \check{D} \\ \| & & \| \\ G_{\mathbb{R}}/H & \subset & G_{\mathbb{C}}/B \end{array} \qquad (H = G_{\mathbb{R}} \cap B)$$

where B is a parabolic subgroup of $G_{\mathbb{C}}$ and H is a compact subgroup of $G_{\mathbb{R}}$. In the case of elliptic curves, $D \subset \check{D}$ is the upper-half-plane $z = x + iy$, $y > 0$ embedded in $\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}$. The group $G_{\mathbb{C}}$ is the group of linear fractional transformations

$z \to az + b/(cz + d)$, $G_{\mathbb{R}}$ is the subgroup of real transformations, and D is the $G_{\mathbb{R}}$ orbit of $\sqrt{-1}$. Since $\Gamma \subset G_{\mathbb{Z}}$, the monodromy group is a discrete subgroup of $G_{\mathbb{R}}$ and acts properly discontinuously on D. Consequently, D/Γ is an analytic space and the period mapping is a holomorphic mapping

$$\Omega: \ S \to D/\Gamma.$$

In the case of curves, D is biholomorphic to a bounded domain in $\mathbb{C}^{g(g+1)/2}$. However, for $n > 1$, D is no longer a bounded domain and consequently the holomorphic mappings into D will *not* have the strong function-theoretic properties (e. g. normal families) which are present when D is a bounded domain. However, if we consider only the mappings into D which satisfy the infinitesimal bilinear relation (III), then it is increasingly becoming clearer that these have the qualitative properties of mappings into a bounded domain. Thus, e. g., a holomorphic mapping

$$\Phi: \ \Delta^* \to D$$

of the punctured disc $0 < |t| < 1$ in D which satisfies (III) will extend continuously across $t = 0$. A much deeper recent result is due to Wilfried Schmid, who has proved that an arbitrary holomorphic mapping

$$\Phi: \ \Delta^* \to D/G_{\mathbb{Z}}$$

which satisfies (III) is, when $|t| \to 0$, strongly asymptotic to an orbit

$$\exp\left(\frac{\log t}{2\pi\sqrt{-1}} N\right)\Omega_0$$

where N is a very special nilpotent transformation of $E_{\mathbb{Z}}$ and Ω_0 is a point in D. From this it follows that the asymptotic analysis of these periods of algebraic integrals is reduced to a problem in Lie groups.

3. Deeper properties and open questions concerning the period mapping.

We want to discuss the analogues of the properties (*a*)-(*d*) for the periods of the elliptic curve in the general case of a period mapping

$$\Omega: \ S \to D/\Gamma$$

arising from an algebraic family $\{ V_s \}_{s \in S}$ of algebraic varieties.

(*a*) Of course the point $\Omega(s) \in D/\Gamma$ depends only on the intrinsic structure of V_s. However, except for curves there is essentially nothing general known about the global equivalence relation determined by Ω. There is some heuristic evidence that, in general, the equivalence relation might be closely related to birational equivalence; i. e. the " Torelli property " should hold in general. Along these lines, it is perhaps an easier problem to determine the equivalence relation infinitesimally; i. e. to find the kernel of the differential $d\Omega$. The best example known here seems to be when the V_s are smooth hypersurfaces in projective space. Then, except for the obvious example of cubic surfaces, the differential $d\Omega$ is injective on the biregular moduli space of the V_s (" local Torelli property ").

The dual problem to finding the equivalence relation of Ω is to determine which points of D come from algebraic varieties. When D is the Siegel upper-half-plane, even though not every point $\Omega \in D$ is the period matrix of a curve, it is obviously the case that every Ω is the period matrix of an abelian variety and therefore may be said to come from algebraic geometry. However, this is essentially the only case when all points are a period matrix of some algebraic variety, and to my knowledge there is not yet even a plausible candidate for the set of points in D which arise from algebraic geometry.

(*b*) Concerning the " size " of the monodromy group Γ, we have Deligne's theorem that Γ is semi-simple and the result that the image $\Omega(S)$ has finite volume in D/Γ. From this it follows that if Γ' is any larger discrete subgroup of $G_{\mathbb{R}}$ which leaves invariant the inverse image $\pi^{-1}(\overline{\Phi(S)})$ for $\pi : D \to D/\Gamma$ the projection, then Γ is of finite index in Γ'. These facts, plus a few examples, indicate that it might be the case that there is a semi-simple subgroup $G'_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$ such that the monodromy group is commensurable with $G'_{\mathbb{Z}} = G_{\mathbb{Z}} \cap G'_{\mathbb{Q}}$ (recall that this means that $\Gamma \cap G'_{\mathbb{Z}}$ is of finite index in both Γ and $G'_{\mathbb{Z}}$). The available evidence certainly indicates that Γ should be large.

(*c*) Matters are somewhat better regarding the " rigidity property ", which states that the period mapping $\Omega : S \to D/\Gamma$ is determined by its value at one point together with the induced map $\Omega_* : \pi_1(S) \to \Gamma$. This property was proved by myself for an arbitrary holomorphic mapping Ω satisfying the infinitesimal bilinear relation (III) but making the strong assumption that S is complete. Then Deligne proved the result in case Ω arises from a family $\{ V_s \}_{s \in S}$ of algebraic varieties. The result for a general holomorphic mapping Ω satisfying (III) follows from Schmid's nilpotent orbit theorem mentioned above.

(*d*) Given a period mapping Ω: $S \to D/\Gamma$, it is expected that the equivalence relation given by Ω is at least an algebraic equivalence relation; i. e. there should exist a sub-field $\mathscr{R}_\Omega$ of the field $\mathscr{R}$ of rational functions on S such that $\Omega(s) = \Omega(s')$ if, and only if, $\psi(s) = \psi(s')$ for all $\psi \in \mathscr{R}_\Omega$. Furthermore, by analogy with the classical case $n = 1$, it is to be hoped that $\mathscr{R}_\Omega$ arises by composing the mapping Ω with something on D/Γ. More precisely, we should like it to be the case that the discrete series representations in $L^2(G_{\mathbb{R}})$ lead to the construction of some " analytic objects " on D/Γ which, upon composition with Ω, yield $\mathscr{R}_\Omega$. This is a problem of fundamental importance, which may well be related to the question mentioned above of saying which points of D come from algebraic geometry, and about which nothing really is known. What is known is that the discrete series part of $L^2(G_{\mathbb{R}})$ seems to lead to " automorphic cohomology " on D/Γ, but it is a mystery as to what this might have to do with algebraic geometry.

These problems mentioned here are discussed in more details in the survey paper referred to at the beginning of this talk. This survey paper also contains some conjectures not discussed above as well as the references for all of the material presented.

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