

**THE VOLUME OF THE FUNDAMENTAL DOMAIN FOR SOME
ARITHMETICAL SUBGROUPS OF CHEVALLEY GROUPS**

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Let $\mathfrak{g}_{\mathbf{Q}}$ be a split semisimple Lie algebra of linear transformations of the finite-dimensional vector space $V_{\mathbf{Q}}$ over \mathbf{Q} . Let $\mathfrak{h}_{\mathbf{Q}}$ be a split Cartan subalgebra of $\mathfrak{g}_{\mathbf{Q}}$ and choose for each root α of $\mathfrak{h}_{\mathbf{Q}}$ a root vector X_{α} so that if $[X_{\alpha}, X_{-\alpha}] = H_{\alpha}$ then $\alpha(H_{\alpha}) = 2$ and so that there is an automorphism θ of $\mathfrak{g}_{\mathbf{Q}}$ with $\theta(X_{\alpha}) = -X_{-\alpha}$. Let L be the set of weights of $\mathfrak{h}_{\mathbf{Q}}$ and if $\lambda \in L$ let

$$V_{\mathbf{Q}}(\lambda) = \{ v \in V_{\mathbf{Q}} \mid Hv = \lambda(H)v \text{ for all } H \in \mathfrak{h}_{\mathbf{Q}} \};$$

let H_1, \dots, H_p be a basis over \mathbf{Z} of

$$\{ H \mid \lambda(H) \in \mathbf{Z} \text{ if } V_{\mathbf{Q}}(\lambda) \neq 0 \}.$$

As usual, there is associated to $\mathfrak{g}_{\mathbf{Q}}$ a connected algebraic group $G_{\mathbf{C}}$ of linear transformations of $V_{\mathbf{C}} = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$. If H is some lattice in $V_{\mathbf{Q}}$ satisfying

- (i) $M = \sum_{\lambda \in L} M \cap V(\lambda)$,
- (ii) $(X_{\alpha}^n/n!)M \subseteq M$ for all α ,

then we let $G_{\mathbf{Z}} = \{ g \in G_{\mathbf{C}} \mid gM = M \}$. Let ω be a left invariant form on $G_{\mathbf{R}}$ of highest degree which takes the value ± 1 on $(\bigwedge_{i=1}^p H_i) \wedge (\bigwedge_{\alpha > 0} X_{\alpha})$ and let $[dg]$ be the Haar measure associated to ω . Our purpose now is to show the following:

If $\zeta(\cdot)$ is the Riemann zeta function, $\prod_{i=1}^p (t^{2a_i-1} + 1)$ is the Poincaré polynomial of $G_{\mathbf{C}}$, and c is the order of the fundamental group of $G_{\mathbf{C}}$ then

$$\int_{G_{\mathbf{Z}}/G_{\mathbf{R}}} [dg] = c \prod_{i=1}^p \zeta(a_i).$$

The method to be used to find the volume of $G_{\mathbf{Z}} \backslash G_{\mathbf{R}}$ is not directly applicable to $[dg]$. So it is necessary to introduce another Haar measure on the group $G_{\mathbf{R}}$. Let U be the connected subgroup of $G_{\mathbf{C}}$ whose Lie algebra is spanned over \mathbf{R} by

$$\{ X_{\alpha} - X_{-\alpha}, i(X_{\alpha} + X_{-\alpha}), iH_{\alpha} \mid \alpha \text{ a root} \}$$

and let $K = G_{\mathbf{R}} \cap U$. Choose an order on the roots and let $N = N_{\mathbf{R}}$ be the set of real points on the connected algebraic subgroup of $G_{\mathbf{C}}$ with the Lie algebra $\sum_{\alpha > 0} \mathbf{C}X_{\alpha}$. Let $A_{\mathbf{R}}$ be the normalizer of $\mathfrak{h}_{\mathbf{C}}$ in $G_{\mathbf{R}}$. Let dn be the Haar measure on N defined by a form which takes the value ± 1 on $\bigwedge_{\alpha > 0} X_{\alpha}$ and let da be the Haar measure on $A_{\mathbf{R}}$ defined by a form which takes the value ± 1 on $\bigwedge_{i=1}^p H_i$. Let dk be the Haar measure on K such that the total volume of K is one. Let $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$

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and let $\xi_{2\rho}(a)$ be the character of $A_{\mathbf{C}}$ associated to 2ρ . Finally let dg be such that

$$\int_{G_{\mathbf{R}}} \phi(g) dg = \int_{N \times A_{\mathbf{R}} \times K} |\xi_{2\rho}(a)|^{-1} \phi(nak) dn da dk.$$

If N^- is the set of real points on the group associated to $\sum_{\alpha < 0} \mathbf{C}X_{\alpha}$ define dn^- in the same way as we defined dn . It is easy to see that

$$\int_G \phi(g) [dg] = \int_N dn \int_{A_{\mathbf{R}}} da \int_{N^-} dn^- |\xi_{2\rho}(a)|^{-1} \phi(nan^-).$$

Suppose $\phi(gk) = \phi(g)$ for all $g \in G_{\mathbf{R}}$ and all $k \in K$. Then

$$\int_G \phi(g) dg = \int_{N \times A_{\mathbf{R}}} dn da |\xi_{2\rho}(a)|^{-1} \phi(na).$$

On the other hand, if $n^- = n(n^-)a(n^-)k(n^-)$,

$$\begin{aligned} \int_G \phi(g) [dg] &= \int_{N^-} dn^- \left\{ \int_A da \int_N dn |\xi_{2\rho}(a)|^{-1} \phi(nan(n^-)a(n^-)k(n^-)) \right\} \\ &= \left\{ \int_A da \int_N dn |\xi_{2\rho}(a)|^{-1} \phi(na) \right\} \left\{ \int_{N^-} |\xi_{2\rho}a(n^-)| dn^- \right\}. \end{aligned}$$

It follows from a formula of Gindikin and Karpelevich that the second factor equals

$$\begin{aligned} \prod_{\alpha > 0} \frac{\pi^{-1/2} \Gamma(\rho(H_{\alpha})/2)}{\Gamma((\rho(H_{\alpha}) + 1)/2)} &= \prod_{\alpha > 0} \frac{\pi^{-\rho(H_{\alpha})/2} \Gamma(\rho(H_{\alpha})/2)}{\pi^{-(\rho(H_{\alpha})+1)/2} \Gamma((\rho(H_{\alpha}) + 1)/2)} \\ &= \frac{\prod'_{\alpha > 0} \pi^{-\rho(H_{\alpha})/2} \Gamma(\rho(H_{\alpha})/2)}{\prod_{\alpha > 0} \pi^{-(\rho(H_{\alpha})+1)/2} \Gamma((\rho(H_{\alpha}) + 1)/2)}, \end{aligned}$$

since when α is simple $\rho(H_{\alpha}) = 1$ and

$$\pi^{-1/2} \Gamma\left(\frac{1}{2}\right) = 1.$$

The product in the numerator is taken over the positive roots which are not simple. By a well-known result the numbers, with multiplicities, in the set

$$\{\rho(H_{\alpha}) + 1 \mid \alpha > 0\}$$

are just the numbers $\rho(H_{\alpha})$ with α positive and not simple, together with the numbers a_1, \dots, a_p . So if

$$\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

we have to show that

$$\int_{G_{\mathbf{Z}}/G_{\mathbf{R}}} dg = \frac{c \prod_{\alpha > 0} \xi(\rho(H_{\alpha}) + 1)}{\prod'_{\alpha > 0} \xi(\rho(H_{\alpha}))}.$$

By the way, it is well to keep in mind that $\rho(H_{\alpha}) > 1$ if α is not simple.

Let A be the connected component of $A_{\mathbf{R}}$ and let M be the points of finite order in $A_{\mathbf{R}}$. Certainly $A_{\mathbf{R}} = AM$. Moreover, by Iwasawa, $G = NAK$. If $g = nak$ and $a = \exp H$, we set $H = H(g)$.

If ϕ is an infinitely differentiable function with compact support on $N \backslash G$ such that $\phi(gk) = \phi(g)$ for all g in G and all k in K we can write ϕ as a Fourier integral.

$$\phi(g) = \frac{1}{(2\pi)^p} \int_{\operatorname{Re} \lambda = \lambda_0} \exp\left(\lambda(H(g)) + \rho(H(g))\right) \Phi(\lambda) |d\lambda|;$$

λ is the symbol for an element of the dual of $\mathfrak{h}_{\mathbf{C}}$; $\Phi(\lambda)$ is an entire complex-valued function of λ ; and $d\lambda = dz_1 \wedge \cdots \wedge dz_p$ with $z_i = \lambda(H_i)$. As in the lectures on Eisenstein series we can introduce

$$\widehat{\phi}(g) = \sum_{\gamma \in G_{\mathbf{Z}} \cap NM \backslash G_{\mathbf{Z}}} \phi(\gamma g).$$

Our evaluation of the volume of $G_{\mathbf{Z}} \backslash G_{\mathbf{R}}$ will be based on the simple relation

$$(\widehat{\phi}, 1)(1, \widehat{\psi}) = (1, 1)(\Pi \widehat{\phi}, \Pi \widehat{\psi}).$$

The inner products are taken in $L^2(G_{\mathbf{Z}} \backslash G_{\mathbf{R}})$ with respect to dg and Π is the orthogonal projection on the space of constant functions. Since

$$(1, 1) = \int_{G_{\mathbf{Z}} \backslash G_{\mathbf{R}}} dg,$$

it is enough to find an explicit formula for the other three terms. Now

$$\begin{aligned} (\widehat{\phi}, 1) &= \int_{G_{\mathbf{Z}} \cap NM \backslash G_{\mathbf{R}}} \phi(g) dg \\ &= \mu(G_{\mathbf{Z}} \cap NM \backslash NM) \int_A |\xi_{2\rho}(a)|^{-1} \phi(a) da \\ &= \Phi(\rho) \end{aligned}$$

since $\mu(G_{\mathbf{Z}} \cap NM \backslash NM) = 1$. To see the latter we have to observe that $M \subseteq G_{\mathbf{Z}}$ and that, as follows from results stated in Cartier's talk, $\mu(G_{\mathbf{Z}} \cap N \backslash N) = 1$. It is also clear that $(1, \widehat{\psi}) = \overline{\Psi}(\rho)$. The nontrivial step is to evaluate

$$(\Pi \widehat{\phi}, \Pi \widehat{\psi}).$$

From the theory of Eisenstein series we know that

$$(\widehat{\phi}, \widehat{\psi}) = \frac{1}{(2\pi)^p} \int_{\operatorname{Re} \lambda = \lambda_0} \sum_{s \in \Omega} M(s, \lambda) \Phi(\lambda) \overline{\Psi}(-s\bar{\lambda}) |d\lambda|.$$

Ω is the Weyl group, λ_0 is any point such that $\lambda_0(H_\alpha) > 1$ for every simple root, and

$$M(s, \lambda) = \prod_{\alpha > 0} \frac{\xi(1 + s\lambda(H_\alpha))}{\xi(1 + \lambda(H_\alpha))} = \prod_{\substack{\alpha > 0 \\ s\alpha < 0}} \frac{\xi(\lambda(H_\alpha))}{\xi(1 + \lambda(H_\alpha))}.$$

In the lectures on Eisenstein series I introduced an unbounded self-adjoint operator A on the closed subspace of $L^2(G_{\mathbf{Z}} \backslash G_{\mathbf{R}})$ generated by the functions $\widehat{\phi}$ with ϕ of the form indicated above. Comparing the definition of A with the formula for $(\widehat{\phi}, 1)$ we see that

$$(A\widehat{\phi}, 1) = (\rho, \rho)(\widehat{\phi}, 1),$$

since the constant functions are in this space $A1 = (\rho, \rho) \cdot 1$. As a consequence, if $E(x)$, $-\infty < x < \infty$, is the spectral resolution of A the constant functions are in the range of $E((\rho, \rho)) - E((\rho, \rho) - 0) = E$. We show that this range consists precisely of the constant functions and compute $(E\widehat{\phi}, \widehat{\psi}) = (\Pi \widehat{\phi}, \Pi \widehat{\psi})$.

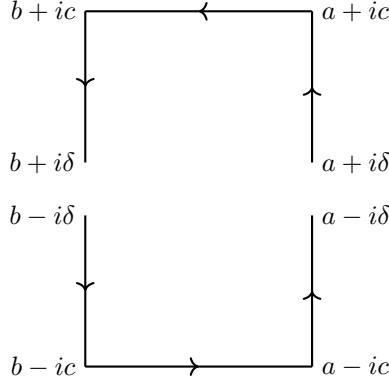
Suppose $a > (\rho, \rho) > b$ and $a - b$ is small. According to a well-known formula,

$$\frac{1}{2} \left\{ \left(E(a) \widehat{\phi}, \widehat{\psi} \right) + \left(E(a-0) \widehat{\phi}, \widehat{\psi} \right) \right\} - \frac{1}{2} \left\{ \left(E(b) \widehat{\phi}, \widehat{\psi} \right) + \left(E(b-0) \widehat{\phi}, \widehat{\psi} \right) \right\}$$

is equal to

$$(a) \quad \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} \left(R(\mu, A) \widehat{\phi}, \widehat{\psi} \right) d\mu$$

if $C(a, b, c, \delta)$ is the following contour.



Recall that, if $\operatorname{Re} \mu > (\lambda_0, \lambda_0)$,

$$\left(R(\mu, A) \widehat{\phi}, \widehat{\psi} \right) = \sum_{s \in \Omega} \frac{1}{(2\pi i)^p} \int_{\operatorname{Re} \lambda = \lambda_0} \frac{1}{\mu - (\lambda, \lambda)} M(s, \lambda) \Phi(\lambda) \overline{\Psi}(-s\bar{\lambda}) d\lambda.$$

If $w = (w_1, \dots, w_p)$ belongs to \mathbf{C}^p let $\lambda(w)$ be such that $\lambda(H_{\alpha_i}) = w_i$, where $\alpha_1, \dots, \alpha_p$ are the simple roots. Set

$$\begin{aligned} \phi_p(w, s) &= M(s, \lambda(w)) \Phi(\lambda(w)) \overline{\Psi}(-s\lambda(\bar{w})), \\ Q_p(w) &= (\lambda(w), \lambda(w)), \end{aligned}$$

then (a) is equal to

$$\frac{1}{c} \sum_{s \in \Omega} \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} d\mu \left\{ \frac{1}{(2\pi i)^p} \int_{\operatorname{Re} w = w_0} \frac{1}{\mu - Q_p(w)} \phi_p(w, s) dw_1 \cdots dw_p \right\}$$

provided each of these limits exist.¹ The coordinates of w_0 must all be greater than one. We shall consider the limits individually.

Let $w^q = (w_1, \dots, w_q)$ and define $\phi_q(w^q; s)$ inductively for $0 \leq q \leq p$ by

$$\phi_q(w_1, \dots, w_q; s) = \operatorname{Residue}_{w_{q+1}=1} \phi_{q+1}(w_1, \dots, w_{q+1}; s).$$

It is easily seen that $\phi_q(w^q; s)$ has no singularities in the region defined by the inequalities $\operatorname{Re} w_i > 1$, $1 \leq i \leq q$; that $\phi_q(w^q; s)$ goes to zero very fast when the imaginary part of w^q goes to infinity and its real part remains in a compact subset of this region; and that there is a positive number ϵ such that the only singularities of $\phi_q(w^q; s)$ in

$$\left\{ (w_1, \dots, w_q) \mid |\operatorname{Re} w_i - 1| < \epsilon, 1 \leq i \leq q \right\}$$

¹The inner integral is defined for $\operatorname{Re} \mu > Q_p(w_0)$. However, as can be seen from the discussion to follow, the function of μ it defines can be analytically continued to a region containing $C(a, b, c, \delta)$.

lie on the hyperplanes $w_i = 1$ and are at most simple poles. $\phi_0(s)$ is of course a constant. Set $Q_q(w^q) = Q_p(w_1, \dots, w_q, 1, \dots, 1)$.

Let us show by induction that the given limit equals

$$(b) \quad \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} d\mu \left\{ \frac{1}{(2\pi i)^q} \int_{\text{Re } w^q = w_0^q} \frac{1}{\mu - Q_q(w^q)} \phi_q(w^q; s) dw_1 \cdots dw_q \right\}$$

if $w_0^q = (w_{0,1}, \dots, w_{0,q})$ with $w_{0,i} > 1$, $1 \leq i \leq q$. Of course, the above expression is independent of the choice of such a point w_0^q . Take $w_0^q = (1 + u, \dots, 1 + u, 1 + v)$, with u and v positive but small and $w_0^{q-1} = (1 + u, \dots, 1 + u)$. If $\Lambda_1, \dots, \Lambda_q$ are such that $\Lambda_i(H_{\alpha_j}) = \delta_{ij}$, then $(\Lambda_i, \Lambda_j) \geq 0$. As a consequence, if u is much smaller than v , then

$$Q_q(1 + u, \dots, 1 + u, 1 + v) < (\rho, \rho).$$

Choose (b) to be larger than the number on the left. Also

$$\text{Re } Q_q(w^q) = Q_q(\text{Re } w^q) - Q_p(\text{Im } w_1, \dots, \text{Im } w_q, 0, \dots, 0).$$

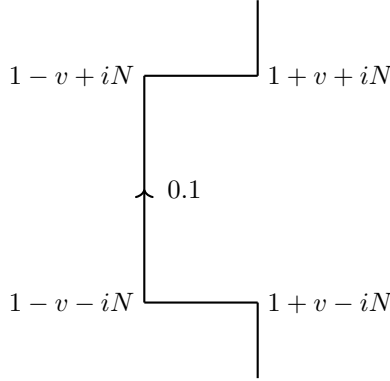
Thus there is a constant N such that if either $\text{Re } w_i = 1 + u$, $1 \leq i \leq q - 1$ and $\text{Re } w_q = 1 + v$ or $\text{Re } w_i = 1 + u$, $1 \leq i \leq p$ and $|\text{Re } w_q - 1| \leq v$ and $|\text{Im } w_q| > N$, then

$$\text{Re } Q_q(w^q) < b - 1/N.$$

In (b) we may perform the integrations in any order. Integrate first with respect to w_q . If C is the indicated contour, the result is the sum of (b) with q replaced by $q - 1$ and

$$\lim_{\delta \downarrow 0} \frac{1}{(2\pi i)^q} \int_{\text{Re } w^{q-1} = w_0^{q-1}} dw_1 \cdots dw_{q-1} \int_C dw_q \phi_q(w^q, s) \left\{ \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} \frac{1}{\mu - Q_q(w^q)} d\mu \right\}$$

which is obviously zero.



The contour C

Taking $q = 0$ in (b) we get

$$\lim_{\delta \downarrow 0} \frac{\phi_0(s)}{2\pi i} \int_{C(a,b,c,\delta)} \frac{1}{\mu - (\rho, \rho)} d\mu = \phi_0(s).$$

It is clear that $\phi_0(s)$ is zero unless s sends every positive root to a negative root but that for the unique element of the Weyl group which does this

$$\phi_0(s) = \frac{\prod'_{\alpha>0} \xi(\rho(H_\alpha)) \Phi(\rho) \overline{\Psi(\rho)}}{\prod_{\alpha>0} \xi(\rho(H_\alpha) + 1)}$$

since $s\rho = -\rho$. This is the result required.

Finally, I remark that although the method just described for computing the volume of $\Gamma \backslash G$ has obvious limitations, it can be applied to other groups. In particular it works for Chevalley groups over a number field.

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